ELEMENTS OF ALGEBRA,

BY

S. F. LACROIX.

TRANSLATED FROM THE FRENCH

FOR THE USE OF THE STUDENTS OF THE UNIVERSITY

AT

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ADVERTISEMENT.

Lacroix's Algebra has been in use in the French schools for a considerable time. It has been approved by the best judges, and been generally preferred to the other elementary treatises, which abound in France. The following translation is from the eleventh edition, printed at Paris in 1815. No alteration has been made from the original, except to substitute English instead of French measures in the questions, where it was thought necessary. When there has been an occasion to add a note by way of illustration, the reference is made by a letter or an obelisk, the author's being always distinguished by an asterisk.

In a review of the two first parts of the Cambridge course of Mathematics, which appeared in the *American Journal of Science and the Arts* for 1822, after many favourable remarks, the writer, speaking of Lacroix's Algebra, observes, that "there are instances of incorrect translation at pages 18, 23, 54." It is regretted that the passages referred to were not more particularly pointed out. The places mentioned, however, have been carefully examined and compared with the original. At page 18 the only passage to which the above remark can be supposed to apply, is the following; "and by arranging the letters in alphabetical order, they are more easily read;" of which the original reads thus:

"et en intervertissant l'ordre des multiplications pour conserver l'ordre alphabétique, plus facile dans l'énonciation des lettres."

Here, as in other parts, a little latitude is used for the sake of perspicuity, and of preserving the English idiom; but it is presumed that the sense is fully and exactly rendered. At page 23 there was clearly a mistake, the sense being the reverse of that of the original, and of that which the connexion obviously requires. At page 54, the only inaccuracy to be found is in printing "multiplier" for "multiple." "At page 37" [97], says the

reviewer, "the last clause, 'and retaining the accents which belonged to the coefficients,' does not express the meaning of the original." The original of the whole passage runs thus;

"en changeant le coefficient de l'inconnue qu'on cherche, dans le terme tont connu, et en conservant d'ailleurs les accens tels qu'ils sont."

It is not easy to perceive in what the defect of the translation consists. A literal rendering would not be very good English; moreover, there is an ambiguity in the original which does not exist in the translation. A doubt might arise in the mind of the learner which accents are meant, those which belong to the terms changed, or those which belong to the terms into which the change is made. In the translation the sense is precise, correct, and clear. Speaking of explanatory notes, the reviewer says, "in that given at page 95, doubtless by inadvertence, the parentheses, which ought to indicate the multiplication between the factors, are omitted." Parentheses in this case would be superfluous, the line separating the numerator from the denominator answering that purpose. In proof of this, examples might be quoted from writers of the first authority. Thus, page 82 of this very work, we have $c - b\,\frac{af - cd}{ae - bd}$, perfectly similar to the case in question, and which is represented as faulty.

Cambridge, July, 1825.

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ELEMENTS OF ALGEBRA.

Preliminary Remarks upon the Transition from Arithmetic to Algebra—Explanation and Use of Algebraic Signs.

1. It must have been remarked in the *Elementary Treatise of Arithmetic*, that there are many questions, the solution of which is composed of two parts; the one having for its object to find to which of the four fundamental rules the determination of the unknown number by means of the numbers given belongs, and the other the application of these rules. The first part, independent of the manner of writing numbers, or of the system of notation, consists entirely in the development of the consequences which result directly or indirectly from the enunciation, or from the manner in which that which is enunciated connects the numbers given with the numbers required, that is to say, from the relations which it establishes between these numbers. If these relations are not complicated, we can for the most part find by simple reasoning the value of the unknown numbers. In order to this it is necessary to analyze the conditions, which are involved in *the relations enunciated*, by reducing them to a course of equivalent expressions, of which the last ought to be one of the following; *the unknown quantity equal to the sum or the difference, or the product, or the quotient, of such and such magnitudes.* This will be rendered plainer by an example.

To divide a given number into two such parts, that the first shall exceed the second by a given difference.

In order to this we would observe 1, that,

The greater part is equal to the less added to the given excess, and that by consequence, if the less be known, by adding to it this excess we have the greater; 2, that,

The greater added to the less forms the number to be divided.

Substituting in this last proposition, instead of the words, *the greater part*, the equivalent expression given above, namely, *the less part added to the given excess*, we find that

The less part, added to the given excess, added moreover to the less part, forms the number to be divided.

But the language may be abridged, thus,

Twice the less part, added to the given excess, forms the number to be divided;

whence we infer, that,

Twice the less part is equal to the number to be divided diminished by the given excess;

and that,

Once the less part is equal to half the difference between the number to be divided and the given excess.

Or, which is the same thing,

The less part is equal to half the number to be divided, diminished by half the given excess.

The proposed question then is resolved, since to obtain the parts sought it is sufficient to perform operations purely arithmetical upon the given numbers.

If, for example, the number to be divided were 9, and the excess of the greater above the less 5, the less part would be, according to the above rule, equal to $\frac{9}{2}$ less $\frac{5}{2}$, or $\frac{4}{2}$, or 2; and the greater, being composed of the less plus the excess 5, would be equal to 7.

2. The reasoning, which is so simple in the above problem, but which becomes very complicated in others, consists in general of a certain number of expressions, such as *added to*, *diminished by*, *is equal to*, &c. often repeated. These expressions relate to the operations by which the magnitudes, that enter into the enunciation of the question, are connected among themselves, and it is evident, that the expressions might be abridged by representing each of them by a sign. This is done in the following manner.

To denote addition we use the sign $+$, which signifies *plus*.

For subtraction we use sign $-$, which signifies *minus*.

For multiplication we use the sign $\times$, which signifies *multiplied by*.

To denote that two quantities are to be divided one by the

other, we place the second under the first with a straight line between them; $\frac{5}{4}$ signifies 5 *divided by* 4.

Lastly, to indicate that two quantities are equal, we place between them the sign $=$ which signifies *equal*.

These abbreviations, although very considerable, are still not sufficient, for we are obliged often to repeat *the number to be divided*, *the number given*, *the less part*, *the number sought*, &c. by which the process is very much retarded.

With respect to given quantities, the expedient which first offers itself is, to take for representing them determinate numbers, as in arithmetic, but this not being possible with respect to the unknown quantities, the practice has been to substitute in their stead a conventional sign, which varies as occasion requires. We have agreed to employ the letters of the alphabet, generally using the last; as in arithmetic we put x for the fourth term of a proportion, of which only the three first are known. It is from the use of these several signs that we derive the science of *Algebra*.

I now proceed by means of them to consider the question stated above (1). I shall represent the unknown quantity, or the less number, by the letter x, for example, the number to be divided and the given excess by the two numbers 9 and 5; the greater number, which is sought, will be expressed by $x + 5$, and the sum of the greater and less by $x + 5 + x$; we have then

$$x + 5 + x = 9;$$

but by writing $2x$ for twice the quantity x there will result

$$2x + 5 = 9.$$

This expression shows that 5 must be added to the number $2x$ to make 9, whence we conclude that

$$2x = 9 - 5,$$

or that $$2x = 4,$$

and that lastly $$x = \frac{4}{2} = 2.$$

By comparing now the import of these abridged expressions, which I have just given by means of the usual signs, with the process of simple reasoning, by which we are led to the solution, we shall see that the one is only a translation of the other.

The number 2, the result of the preceding operations, will answer only for the particular example which is selected, while the course of reasoning considered by itself, by teaching us, that

the less part is equal to half the number to be divided, minus half the given excess, renders it evident, that the unknown number is composed of the numbers given, and furnishes a rule by the aid of which we can resolve all the particular cases comprehended in the question.

The superiority of this method consists in its having reference to no one number in particular; the numbers given are used throughout without any change in the language by which they are expressed; whereas, by considering the numbers as determinate, we perform upon them, as we proceed, all the operations which are represented, and when we have come to the result there is nothing to show, how the number 2, to which we may arrive by any number of different operations, has been formed from the given numbers 9 and 5.

3. These inconveniences are avoided by using characters to represent the number to be divided and the given excess, that are independent of every particular value, and with which we can therefore perform any calculation. The letters of the alphabet are well adapted to this purpose, and the proposed question by means of them may be enunciated thus,

To divide a given number represented by a *into two such parts that the greater shall have with respect to the less a given excess represented by* b.

Denoting always the less by x;

The greater will be expressed by $x + b$;

Their sum, or the number to be divided, will be equal to $x + x + b$, or $2x + b$;

The first condition of the question then will give

$$2x + b = a.$$

Now it is manifest that, if it is necessary to add to double of x, or to $2x$, the quantity b in order to make the quantity a, it will follow from this, that it is necessary to diminish a by b to obtain $2x$, and that consequently $2x = a - b$.

We conclude then that half of $2x$ or $x = \frac{a}{2} - \frac{b}{2}$.

This last result, being translated into ordinary language, by substituting the words and phrases denoted by the letters and signs which it contains, gives the rule found before, according to which, *in order to obtain the less of two parts sought we subtract*

from half of the number to be divided, or from $\frac{a}{2}$ *half of the given excess, or* $\frac{b}{2}$.

Knowing the less part we have the greater by adding to the less the given excess. This remark is sufficient for effecting the solution of the question proposed; but Algebra does more; it furnishes a rule for calculating the greater part without the aid of the less as follows;

$\frac{a}{2} - \frac{b}{2}$ being the value of this, augmenting it by the excess b, we have for the greater part $\frac{a}{2} - \frac{b}{2} + b$. Now $\frac{a}{2} - \frac{b}{2} + b$ shows that after having subtracted from $\frac{a}{2}$ the half of b, it is necessary to add to the remainder the whole of b, or two halves of b, which reduces itself to augmenting $\frac{a}{2}$ by the half of b, or by $\frac{b}{2}$. It is evident then that $\frac{a}{2} - \frac{b}{2} + b$ becomes $\frac{a}{2} + \frac{b}{2}$; and by translating this expression we learn, that *of the two parts sought the greater is equal to half of the number to be divided plus half of the given excess.*

In the particular question which I first considered, the number to be divided was 9, the excess of one part above the other 5; in order to resolve it by the rules to which we have just arrived, it will be necessary to perform upon the numbers 9 and 5, the operations indicated upon a and b.

The half of 9 being $\frac{9}{2}$ and that of 5 being $\frac{5}{2}$, we have for the less part

$$\tfrac{9}{2} - \tfrac{5}{2} = \tfrac{4}{2} = 2,$$

and for the greater

$$\tfrac{9}{2} + \tfrac{5}{2} = \tfrac{14}{2} = 7.$$

4. I have denoted in the above the less of the two parts by x, and I have deduced from it the greater. If it were required to find directly this last, it should be observed, that representing it by x, the other will be $x - b$, since we pass from the greater to the less by subtracting the excess of the first above the second; the number to be divided will then be expressed by $x + x - b$, or by $2x - b$, and we have consequently $2x - b = a$.

This result makes it evident that $2x$ exceeds the quantity a

by the quantity b, and that consequently $2x = a + b$. By taking the half of $2x$ and of the quantity which is equal to it, we obtain for the value of x

$$x = \frac{a}{2} + \frac{b}{2},$$

which gives the same rule as the above for determining the greater of the two parts sought. I will not stop to deduce from it the expression for the smaller.

The same relation between the numbers given and the numbers required may be enunciated in many different ways. That which has led to the preceding result is deduced also from the following enunciation:

Knowing the sum a *of two numbers and their difference* b, *to find each of those numbers;* since, in other words, the number to be divided is the sum of the two numbers sought, and their difference is the excess of the greater above the less. The change in the terms of the enunciation being applied to the rules found above, we have

The less of two numbers sought is equal to half of the sum minus half of the difference.

The greater is equal to half of the sum plus half of the difference.

5. The following question is similar to the preceding, but a little more complicated.

To divide a given number into three such parts, that the excess of the mean above the least may be a given number, and the excess of the greatest above the mean may be another given number.

For the sake of distinctness I will first give determinate values to the known numbers.

I will suppose that the number to be divided is 230;
that the excess of the middle part above the least is 40; and
that of the greatest above the middle one is 60.

Denoting the least part by x,
the middle one will be the least plus 40, or $x + 40$, and the greatest will be the middle one plus 60, or $x + 40 + 60$.

Now the three parts taken together must make the number to be divided; whence,

$$x + x + 40 + x + 40 + 60 = 230.$$

If the given numbers be united in one expression and the unknown ones in another, x is found three times in the result, and for the sake of conciseness we write

$$3x + 140 = 230.$$

But since it is necessary to add 140 to triple of x to make 230, it follows, that by taking 140 from 230 we have exactly the triple of x, or

$$3x = 230 - 140,$$

or
$$3x = 90,$$

whence it follows that

$$x = \tfrac{90}{3} = 30.$$

By adding to 30 the excess 40 of the middle part above the least, we have 70 for the middle part.

By adding to 70 the excess 60 of the greatest above the middle part, we have 130 for the greatest.

6. If the known numbers were different from those which I have used in the enunciation, we should still resolve the question by following the course pursued in the preceding article, but we should be obliged to repeat all the reasonings and all the operations, by which we have arrived at the number 30, because there is nothing to show how this number is composed of 230, 40, and 60. To render the solution independent of the particular values of numbers, and to show how the value of the unknown quantity is fixed by means of the known quantities, I will enunciate the problem thus;

To divide a given number a *into three such parts, that the excess of the middle one above the least shall be a given number* b, *and the excess of the greater above the middle one shall be a given number* c.

Designating as above by x the unknown quantity and making use of the common signs and the symbols a, b, c, which represent the known quantities in the question, the reasoning already given will be repeated.

The least part $= x$,
the middle part $= x + b$,
the greatest $= x + b + c$.
and the sum of these three makes the number to be divided; hence,

$$x + x + b + x + b + c = a.$$

This expression, which is so simple, may be still further abridged; for since it appears that x enters three times into the number to be divided and b twice, instead of $x + x + x$, I shall write $3x$, and instead of $+ b + b$, I shall write $+ 2b$, and it will become

$$3x + 2b + c = a.$$

From this last expression it is evident, that it is necessary to add to triple the number represented by x, double the number represented by b, and also the number c, in order to make the number a; it follows then, that if from the number a we take double the number b and also the number c, we shall have exactly the triple of x, or that

$$3x = a - 2b - c.$$

Now x being one third of three times x, we thence conclude that

$$x = \frac{a - 2b - c}{3}.$$

It should be carefully observed, that having assigned no particular value to the numbers represented by a, b, c, the result to which we have come is equally indeterminate as to the value of x; it shews merely what operations it is necessary to perform upon these numbers, when a value is assigned to them, in order thence to deduce the value of the unknown quantity.

In short, the expression $\frac{a - 2b - c}{3}$, to which x is equal, may be reduced to common language by writing, instead of the letters, the numbers which they represent, and instead of the signs, the kind of operation which they indicate; it will then become, as follows;

From the number to be divided, subtract double the excess of the middle part above the least, and also the excess of the greatest above the middle part, and take a third of the remainder.

If we apply this rule, we shall determine, by the simple operations of arithmetic, the least part. The number to be divided being for example 230, one excess 40, and the other 60, if we subtract as in the preceding article twice 40, or 80, and 60 from 230, there will remain 90, of which the third part is 30, as we have found already.

If the number to be divided were 520, one excess 50, and the other 120, we should subtract twice 50, or 100, and 120 from 520, and there would remain 300, a third of which or 100 would be the smallest part. The others are found by adding 50 to 100, which makes 150, and 120 more to this, which makes 270, so that the parts sought would be

100, 150, 270,

and their sum would be 520, as the question requires.

It is because the results in algebra are for the most part only an indication of the operations to be performed upon numbers in order to find others, that they are called in general *formulas*.

This question, although more complicated than that of article 1, may still be resolved by ordinary language, as may be seen in the following table, where against each step is placed a translation of it into algebraic characters.

PROBLEM.

To divide a number into three such parts, that the excess of the middle one above the least shall be a given number, and the excess of the greatest above the middle one shall be another given number.

SOLUTION.

By common language.	*By algebraic characters.*
	Let the number to be divided be denoted by a. the excess of the middle part above the least by b. the excess of the greatest above the middle one by c. The least part being x.
The middle part will be the least, plus the excess of the mean above the least.	The middle part will be $x + b$.
The greatest part will be the middle one, plus the excess of the greatest above the middle one. The three parts will together form the number proposed.	The greatest will be $x + b + c$.
Whence the least part, plus the least part, plus the excess of the middle one above the least, plus also the least part, plus the excess of the middle one above the least, plus the excess of the greatest above the middle one, will be equal to the number to be divided.	Whence $x + x + b + x + b + c = a$.

Whence three times the least part, plus twice the excess of the middle part above the least, plus also the excess of the greatest above the middle one, will be equal to the number to be divided.

$$3x + 2b + c = a.$$

Whence three times the least part will be equal to the number to be divided, minus twice the excess of the middle part above the least, and minus also the excess of the greatest above the middle one.

$$3x = a - 2b - c.$$

Whence in fine, the least part will be equal to a third of what remains after deducting from the number to be divided twice the excess of the middle part above the least, and also the excess of the greatest above the middle one.

$$x = \frac{a - 2b - c}{3}.$$

7. The signs mentioned in article 2 are not the only ones used in algebra. New considerations will give rise to others, as we proceed. It must have been observed in article 2, that the multiplication of x by 2, and in articles 5 and 6 that of x by 3 and that of b by 2, is denoted by merely writing the figures before the letters x and b without any sign between them, and I shall express it in this manner hereafter; so that a number placed before a letter is to be considered as multiplied by the number represented by that letter, $5x$, $5a$, &c. signify five times x, five times a, &c. $\frac{3}{4}x$ or $\frac{3x}{4}$, &c. signifies $\frac{3}{4}$ of x or three times x divided by 4, &c.

In general, multiplication will be denoted by writing the factors in order one after the other without any sign between them, whenever it can be done without confusion.

Thus the expressions ax, bc, &c. are equivalent to $a \times x$, $b \times c$, &c., but we cannot omit the sign when numbers are concerned, for then 3×5, the value of which is 15, becomes 35. In this case we often substitute a point in the place of the usual sign, thus, $3 . 5$.

Equations.

8. If the solution of the problems in articles 3 and 6 be examined with attention, it will be found to consist of two parts entirely distinct from each other. In the first place, we express by means of algebraic characters the relations established by the nature of the question between the known and unknown quantities, from which we infer the equality of two quantities among themselves; for instance, in article 3 the quantities $2x + b$ and a, and in article 6 the quantities $3x + 2b + c$ and a.

We afterwards deduce from this equality a series of consequences, which terminate in showing the unknown quantity x to be equal to a number of known quantities connected together by operations, that are familiar to us; this is the second part of the solution.

These two parts are found in almost every problem which belongs to algebra. It is not easy, however, at present to give a rule adapted to the first part, which has for its object to reduce the conditions of the question to algebraic expressions. To be able to do this well, it is necessary to become familiar with the characters used in algebra, and to acquire a habit of analyzing a problem in all its circumstances, whether expressed or implied. But when we have once formed the two numbers, which the question supposes equal, there are regular steps for deducing from this expression the value of the unknown quantity, which is the object of the second part of the solution. Before treating of these I shall explain the use of some terms which occur in algebra.

An *equation* is an expression of the equality of two quantities.

The quantities which are on one side of the sign $=$ taken together are called a *member*; an equation has two *members.*

That which is on the left is called the *first member*, and the other the *second.*

In the equation $2x + b = a$, $2x + b$ is the *first member*, and a is the *second member.*

The quantities, which compose a member, when they are separated by the sign $+$ or $-$, are called *terms.*

Thus, the first member of the equation $2x + b = a$ contains two terms, namely, $2x$ and $+ b$.

The equation $\frac{3}{4}x + 7 = 8x - 12$ has two terms in each of its members, namely,

$\frac{3}{4}x$ and $+ 7$ in the first,
$8x - 12$ in the second.

Although I have taken at random, and to serve for an example merely, the equation $\frac{3}{4}x + 7 = 8x - 12$, it is to be considered, as also every other of which I shall speak hereafter, as derived from a problem, of which we can always find the enunciation by translating the proposed equation into common language. This under consideration becomes,

To find a number x *such, that by adding* 7 *to* $\frac{3}{4}$ x, *the sum shall be equal to* 8 *times* x *minus* 12.

Also the equation $ax + bc - cx = ac - bx$, in which the letters a, b, c, are considered as representing known quantities, answers to the following question;

To find a number x *such, that multiplying it by a given number* a, *and adding the product of two given numbers* b *and* c, *and subtracting from this sum the product of a given number* c *by the number* x, *we shall have a result equal to the product of the numbers* a *and* c, *diminished by that of the numbers* b *and* x.

It is by exercising one's self frequently in translating questions from ordinary language into that of algebra, and from algebra into ordinary language, that one becomes acquainted with this science, the difficulty of which consists almost entirely in the perfect understanding of the signs and the manner of using them.

To deduce from an equation the value of the unknown quantity, or to obtain this unknown quantity by itself in one member and all the known quantities in the other, is called *resolving* the equation.

As the different questions, which are solved by algebra, lead to equations more or less compounded, it is usual to divide them into several kinds of *degrees*. I shall begin with *equations of the first degree*. Under this denomination are included those equations in which the unknown quantities are neither multiplied by themselves nor into each other.

Of the resolution of Equations of the First Degree, having but one unknown quantity.

9. We have already seen that to resolve an equation is to arrive at an expression, in which the unknown quantity alone in

one member is equal to known quantities combined together by operations which are easily performed. It follows then, that in order to bring an equation to this state, it is necessary to *free* the unknown quantity from known quantities with which it is connected. Now the unknown quantity may be united to known quantities in three ways;

1. By addition and subtraction, as in the equations,

$$x + 5 = 9 - x,$$
$$a + x = b - x.$$

2. By addition, subtraction, and multiplication, as in the equations,

$$7x - 5 = 12 + 4x,$$
$$ax - b = cx + d.$$

3. Lastly, by addition, subtraction, multiplication, and division, as in the equations,

$$\frac{5x}{3} + 8 = \frac{11}{12}x + 9,$$
$$\frac{ax}{b} + cx - d = \frac{mx}{n} + \frac{p}{q}.$$

The unknown quantity is freed from additions and subtractions, where it is connected with known quantities, by collecting together into one member all the terms in which it is found; and for this purpose it is necessary to know how to transpose a term from one member to the other.

10. For example, in the equation

$$7x - 5 = 12 + 4x,$$

it is necessary to transpose $4x$ from the second member to the first, and the term -5 from the first member to the second. In order to this, it is obvious, that by cancelling $+4x$ in the second member, we diminish it by the quantity $4x$, and we must make the same subtraction from the first member, to preserve the equality of the two members; we write then $-4x$ in the first member, which becomes $7x - 5 - 4x$ and we have

$$7x - 5 - 4x = 12.$$

To cancel -5 in the first member, is to omit the subtraction of 5 units, or in other words, to augment this member by 5 units; to preserve the equality then we must increase the second member by 5 units, or write $+5$ in this member, which will make it $12 + 5$; we have then

$$7x - 4x = 12 + 5.$$

By performing the operations indicated there will result the equation $$3x = 17.$$

From this mode of reasoning, which may be applied to any example whatever, it is evident, that to cancel in a member a term affected with the sign $+$, which of course augments this member, it is necessary to subtract the term from the other member, or to write it with the sign $-$; that on the contrary when the term to be effaced has the sign minus, as it diminishes the member to which it belongs, it is necessary to augment the other member by the same term, or to write it with the sign $+$; whence we obtain this general rule;

To transpose any term whatever of an equation from one member to the other, it is necessary to efface it in the member where it is found, and to write it in the other with the contrary sign.

To put this rule in practice, we must bear in mind that the first term of each member, when it is preceded by no sign, is supposed to have the sign plus. Thus, in transposing the term cx of the literal equation $ax - b = cx + d$ from the second member to the first, we have

$$ax - b - cx = d;$$

transposing also $-b$ from the first member to the second, it becomes

$$ax - cx = d + b.$$

11. By means of this rule, we can unite together in one of the members all the terms containing the unknown quantity, and in the other all the known quantities; and under this form the member, in which the unknown quantity is found, may always be decomposed into two factors, one of which shall contain only known quantities, and the other shall be the unknown quantity by itself.

This process suggests itself immediately, whenever the proposed equation is numerical and contains no fractions, because then all the terms involving the unknown quantity may be reduced to one. If we have, for example,

$$10x + 7x - 2x = 25 + 7,$$

by performing the operations indicated in each member, we shall have in succession

$$17x - 2x = 32,$$
$$15x = 32;$$

and $15x$ is resolved into two factors 15 and x; we have then

the unknown factor x by dividing the number 32, which is equal to the product $15\,x$ by the given factor 15, thus,

$$x = \tfrac{32}{15}.$$

This resolution is effected in like manner in the literal equations of the form

$$a\,x = b\,c;$$

because the term $a\,x$ signifies the product of a by x; we hence conclude, that

$$x = \frac{b\,c}{a}.$$

Let there be the equation

$$a\,x - b\,x + c\,x = a\,c - b\,c,$$

which contains three terms involving the unknown quantity. Since $a\,x$, $b\,x$, $c\,x$, represent the products respectively of x by the quantities a, b, and c, the expression $a\,x - b\,x + c\,x$ translated into ordinary language is rendered as follows;

From x *taken first, so many times as there are units in* a, *subtract so many times* x *as there are units in* b, *and add to the result the same quantity* x, *taken so many times as there are units in* c.

It follows then on the whole, that the unknown quantity x is taken so many times as there are units in the difference of the numbers a and b, augmented by the number c, that is to say, so many times as is denoted by the number $a - b + c$; the two factors of the first member are therefore $a - b + c$ and x; we have then

$$x = \frac{a\,c - b\,c}{a - b + c}.$$

From this reasoning which may be applied to every other example, it is evident, that *after collecting together into one member the different terms containing the unknown quantity, the factor, by which the unknown quantity is multiplied, is composed of all those quantities by which it is separately multiplied, arranged with their proper signs, and the unknown quantity is found by dividing all the terms of the known member by the factor which is thus obtained.*

According to this rule, the equation $a\,x - 3\,x = b\,c$ gives

$$x = \frac{b\,c}{a - 3}.$$

Also the equation $x + a\,x = c - d$ is reduced to

$$x = \frac{c - d}{1 + a},$$

for it is necessary to observe that the letter x, taken singly, must be regarded as multiplied by one. It is besides manifest, that in $x + ax$, the unknown quantity x is contained once more than in ax, and is consequently multiplied by $1 + a$.

12. It is evident that if there be a factor, which is common to all the terms of an equation, it may be dropped without destroying the equality of the two expressions, since it is merely dividing by the same number all the parts of the two quantities, which are by supposition equal to each other.

Let there be, for example, the equation

$$6abx - 9bcd = 12bdx + 15abc.$$

I observe in the first place, that the numbers 6, 9, 12 and 15 are divisible by 3, and by suppressing this factor, I merely take a third part of all the quantities which compose the equation.

I have after this reduction,

$$2abx - 3bcd = 4bdx + 5abc.$$

I observe, moreover, that the letter b, combined in each term as a multiplier, is a factor common to all the terms; by cancelling it the equation becomes

$$2ax - 3cd = 4dx + 5ac.$$

Applying the rules given in articles 10 and 11, I deduce successively

$$2ax - 4dx = 5ac + 3cd,$$

$$x = \frac{5ac + 3cd}{2a - 4d}.$$

13. I now proceed to equations, the terms of which have divisors. These may be solved by the preceding rules whenever the unknown quantity does not enter into the denominators; but it is often more simple to reduce all the terms to the same denominator which may then be cancelled.

Let there be, for example, the equation

$$\frac{2x}{3} + 4 = \frac{4x}{5} + 12 - \frac{5x}{7}.$$

Arithmetic furnishes rules for reducing fractions to the same denominator, and for converting whole numbers into fractions of a given kind. (*Arith.* 79, 69.) Let all the terms of the proposed equation be transformed by these rules into fractions of the same denominator, beginning with the fractions, which are

$$\frac{2x}{3}, \frac{4x}{5}, \frac{5x}{7}.$$

I convert them by the first of the rules cited into the following;

$$\frac{5\times7\times2x}{3\times5\times7},\qquad\frac{3\times7\times4x}{3\times5\times7},\qquad\frac{3\times5\times5x}{3\times5\times7}.$$

Since, for converting the whole numbers 4 and 12 into fractions, nothing more is necessary than to multiply them by the common denominator of the fractions, namely, $3\times5\times7$; we have

$$3\times5\times7\times4,\qquad3\times5\times7\times12.$$

By placing all these terms in order in the proposed equation, it will become

$$\frac{5\times7\times2x}{3\times5\times7}+\frac{3\times5\times7\times4}{3\times5\times7}$$
$$=\frac{3\times7\times4x}{3\times5\times7}+\frac{3\times5\times7\times12}{3\times5\times7}-\frac{3\times5\times5x}{3\times5\times7}.$$

The denominator may now be cancelled, since by doing it we only multiply all the parts of the equation by this denominator, (*Arith.* 54), which does not destroy the equality of the members. It will become then

$$5\times7\times2x+3\times5\times7\times4$$
$$=3\times7\times4x+3\times5\times7\times12-3\times5\times5x.$$

or $$70x+420=84x+1260-75x,$$

an equation without a denominator from which we deduce the value of x by the preceding rules.

It is evident from inspection, as also from the mere application of the arithmetical rules referred to, that in the above operation *the numerators of each fraction must be multiplied by the product of the denominators of all the others, the whole numbers by the product of all the denominators; then no account need be taken of the common denominators of the fractions thus obtained.*

The equation $70x+420=84x+1260-75x$, becomes successively

$$70x+75x-84x=1260-420,$$
$$61x=840,$$
$$x=\tfrac{840}{61}=13\tfrac{47}{61}.$$

The same process is applicable to literal equations, it being observed, that it is necessary only to indicate the multiplications, which are actually performed when numbers are concerned.

Let there be, for example, the equation

$$\frac{ax}{b}-c=\frac{dx}{e}+\frac{fg}{h};$$

we deduce from it

$$eh\times ax-beh\times c=bh\times dx+be\times fg,$$

a result which may be more simply expressed by placing the factors of each product one after the other without any sign between them, according to the method given in article 7; and by arranging the letters in alphabetical order, they are more easily read, it then becomes

$$aehx - bceh = bdhx + befg,$$

from which is deduced

$$aehx - bdhx = befg + bceh,$$

and
$$x = \frac{befg + bceh}{aeh - bah}.$$

14. Although no general and exact rule can be given for forming the equation of any question whatever; there is notwithstanding, a precept of extensive use, which cannot fail to lead to the proposed object. It is this,

To indicate by the aid of algebraic signs upon the known quantities represented either by numbers or letters, and upon the unknown quantities represented always by letters, the same reasonings and the same operations, which it would have been necessary to perform in order to verify the values of the unknown quantities, had they been known.

In making use of this precept, it is necessary, in the first place, to determine with care what are the operations which are contained in the enunciation of the question, either directly or by implication; but this is the very thing which constitutes the difficulty of *putting a question into an equation.*

The following examples are intended to illustrate the above precept. I have taken the two first from among the questions which are solved by arithmetic, in order to show the advantage of the algebraic method.

1. *Let there be two fountains, the first of which running for $2\frac{1}{2}h.$ fills a certain vessel, and the second fills the same vessel by running $3\frac{3}{4}h.$ what time will be employed by both the fountains running together in filling the vessel?*

If the time were given we should verify it by calculating the quantities of water discharged by each fountain, and adding them together we should be certain, that they would be equal to the whole content of the vessel.

To form the equation we denote the unknown time by x, and we indicate upon x the operations implied by the question; but in order to render the solution independent of the given num-

bers, and at the same time to abridge the expression where fractions are concerned, we will represent them also by letters, a being written instead of $2\frac{1}{2}$h. and b instead of $3\frac{3}{4}$h.

This being supposed, by putting the capacity of the vessel equal to unity, it is evident, that,

The first fountain, which will fill it in a number of hours denoted by a, will discharge into it in one hour a quantity of water expressed by the fraction $\frac{1}{a}$, and that consequently, in a number x of hours it will furnish the quantity $x \times \frac{1}{a}$, or $\frac{x}{a}$. (*Arith.* 53).

The second fountain, which will fill the same vessel in a number of hours described by b, will discharge into it in one hour a quantity of water expressed by the fraction $\frac{1}{b}$, and consequently in a number x of hours, it will furnish the quantity $x \times \frac{1}{b}$, or $\frac{x}{b}$.

The whole quantity of water then furnished by the two fountains, will be

$$\frac{x}{a} + \frac{x}{b};$$

and this quantity must be equal to the content of the vessel, which was considered as unity; we have then the equation

$$\frac{x}{a} + \frac{x}{b} = 1.$$

This equation reduced by the foregoing rules, becomes

$$bx + ax = ab,$$

$$x = \frac{ab}{b+a}.$$

The last formula gives this simple rule for resolving every case of the proposed question.

Divide the product of the numbers, which denote the times employed by each fountain in filling the vessel, by the sum of these numbers; the quotient expresses the time required by both the fountains running together to fill the vessel.

Applying this rule to the particular case under consideration, we have

$$2\tfrac{1}{2} \times 3\tfrac{3}{4} = \tfrac{5}{2} \times \tfrac{15}{4} = \tfrac{75}{8},$$
$$2\tfrac{1}{2} + 3\tfrac{3}{4} = \tfrac{5}{2} + \tfrac{15}{4} = \tfrac{20}{8} + \tfrac{30}{8} = \tfrac{50}{8},$$

whence $$x = \tfrac{75}{50} = \tfrac{3}{2}.$$

2. *Let* a *be a number to be divided into three parts, having among themselves the same ratios as the given numbers* m, n, *and* p.

It is evident that the verification of the question would be as follows;

denoting the 1st part by x, we have

$$m : n :: x : \text{the 2d part} = \frac{nx}{m}, \ (\textit{Arith.}\ 116.)$$

$$m : p :: x : \text{the 3d part} = \frac{px}{m};$$

the three parts added together must make the number to be divided. We have then the equation

$$x + \frac{nx}{m} + \frac{px}{m} = a.$$

By reducing all the terms to the denominator m, it becomes

$$mx + nx + px = am;$$

and we deduce from this

$$x = \frac{am}{m + n + p}.$$

This result is nothing more nor less than an algebraic expression of *the rule of Fellowship*, (*Arith.* 124); for by regarding the numbers m, n, p, as denoting the stocks of several persons trading in company, $m + n + p$ is the whole stock, a the gain to be divided, and the equation

$$x = \frac{ma}{m + n + p}$$

shows that *a share is obtained by multiplying the corresponding stock into the whole gain, and dividing the product by the sum of the stocks;* which reduced to a proportion, becomes

the whole stock : a particular stock
: : the whole gain : to the particular gain.

15. To form an equation from the following question, requires an attention to some things, which have not yet been considered.

A fisherman, to encourage his son, promises him 5 *cents for each throw of the net in which he shall take any fish, but the son, on the other hand, is to remit to the father* 3 *cents for each unsuccessful throw. After* 12 *throws the father and the son settle their account, and the former is found to owe the latter* 28 *cents. What was the number of successful throws of the net?*

If we represent this number by x the number of unsuccessful ones will be $12 - x$; and if these numbers were given, we should verify them by multiplying 5 cents by the first, to obtain what the father was bound to pay the son, and 3 cents by the second, to find what the son engaged to return to the father. The first number ought to exceed the second by 28 cents, which the father owed the son.

We have for the first number x times 5 cents, or $5x$. With respect to the second, there is some difficulty. How are we to obtain the product of 3 by $12 - x$? If instead of x we had a given number, we should first perform the subtraction indicated, and then multiply 3 by the remaïnder; but this cannot be done at present, and we must endeavour to perform the multiplication before the subtraction, or at least, to give the expression an entire algebraic form, similar to that of equations that are readily solved.

With a little attention we shall see, that by taking 12 times the number three, we repeat the number 3 so many times too much, as there are units in the number x, by which we ought first to have diminished the multiplier 12, so that the true product will be 36 diminished by 3 taken x times or $3x$, or more simply $36 - 3x$.

This conclusion may be verified by giving to x a numerical value. If, for example, x were equal to 8, we should have 3 to be taken 12 times — 8 times, and, if we neglect — 8 times, we should make the result 8 times the number 3 too much; the true product then will be

$$3 \times 12 - 3 \times 8 = 36 - 24 = 12.$$

This result agrees with that which would arise from first subtracting 8 from 12; for then

$$12 - 8 = 4, \quad \text{and} \quad 3 \times 4 = 12.$$

This being admitted, since the money due from the father to the son is expressed by $5x$, and that which the son owes the father by $36 - 3x$, the second number must be subtracted from the first in order to obtain the remainder 28; but here is another difficulty; how shall we subtract $36 - 3x$ from $5x$, without having first subtracted $3x$ from 36?

We shall avoid this difficulty by observing, that if we neglect the term $-3x$, and subtract from $5x$ the entire number 36, we shall have taken necessarily $3x$ too much, since it is only what

remains after having diminished 36 by $3x$ that is to be subtracted from $5x$; so that the difference $5x - 36$ ought to be augmented by $3x$ in order to form the quantity that should remain after having taken from $5x$ the number denoted by $36 - 3x$. This quantity will then be

$$5x - 36 + 3x;$$

and we have the equation

$$5x - 36 + 3x = 28,$$

which becomes successively

$$8x - 36 = 28,$$
$$8x = 28 + 36,$$
$$8x = 64,$$
$$x = \tfrac{64}{8} = 8.$$

There have been then 8 successful throws of the net and 4 unsuccessful ones.

Indeed 8 throws at 5 cents a throw give	40 cents,	
4 throws at 3 cents a throw give	12	
difference	28	

as required by the conditions of the question.

To render the solution general, let a represent the sum given by the father to the son for each successful throw of the net, and b the sum returned by the son for each unsuccessful one, and c the total number of throws, and d the sum received on the whole by the son. If x be put equal to the number of successful throws, $c - x$ will express the number of unsuccessful ones; each throw of the former kind being worth to the son a sum a, x throws would be worth $a \times x$ or ax, and the unsuccessful throws would be worth to the father the sum b multiplied by the number $c - x$.

The reasoning by which we have found the parts of the product of 3 by $12 - x$, applies equally to the general case. If we neglect in the first place $-x$ in forming the product bc of b by the whole of c, the sum b will be repeated x times too much, and consequently the true product will be $bc - bx$.

In order to subtract this product from the sum ax, it is necessary to observe, as in the numerical example, that if we subtract the whole of the quantity bc we take the quantity bx too much, by which the former ought to have been first diminished, and that consequently the true remainder is not merely $ax - bc$, but $ax - bc + bx$.

As this sum is equal to d, we have the equation

$$ax - bc + bx = d,$$

which gives

$$ax + bx = d + bc,$$

$$x = \frac{d + bc}{a + b}.$$

As this general formula indicates what operations are to be performed upon the numbers a, b, c, d, in order to obtain the unknown quantity x, we may reduce it to a rule or carefully write instead of the letters a, b, c, d, the numbers given. This last process is called *substituting* the values of the given quantities, or *putting the formula into numbers*. Applying here those of the foregoing example, we have

$$x = \frac{28 + 3 \times 12}{5 + 3};$$

by performing the operations indicated, it becomes

$$x = \frac{28 + 36}{8} = \frac{64}{8} = 8.$$

Methods for performing, as far as is possible, the Operations indicated upon Quantities that are represented by Letters.

16. From the preceding question it is evident, that in certain cases a multiplication indicated upon the sum or difference of several quantities is made to consist of several partial multiplications; and in art. 11, we have exactly the reverse, by resolving the quantity $ax - bx + cx$, which represents the result of several multiplications, followed by additions and subtractions, into the two factors $a - b + c$ and x, which indicate only a single multiplication preceded by addition and subtraction. The reasoning pursued in these two circumstances, will suggest rules for performing, upon quantities represented by letters, operations which are called *algebraic multiplication* and *division*, from the analogy which they have with the corresponding operations of arithmetic.

We have also by the same analogy two algebraic operations, which bear the names of *addition* and *subtraction*, in which the object is to unite several algebraic expressions in one, or to take one expression from another. But these operations, like the preceding, differ from those of arithmetic in this, that their results are, for the most part, only indications of the operations

to be performed; they present only a transformation of the operations originally indicated into others, which produce the same effect. All that is done, is either to simplify the expressions, or to give them a proper form for exhibiting the conditions that are to be fulfilled.

In order to explain these operations, we give the name of *simple quantities* to those which consist only of one term, as $+ 2a$, $- 3ab$, &c. *binomials* to those which consist of two, as $a + b$, $a - b$, $5a - 2x$, &c. *trinomials* to those which consist of three terms, *quadrinomials* to those which consist of four terms, and *polynomials* to those which consist of more than four terms. It may be observed also, that we call polynomials *compound quantities*.

Of the Addition of Algebraic Quantities.

17. The addition of simple quantities is performed by writing them one after the other, with the sign $+$ between them; thus, a added to b is expressed by $a + b$. But when it is proposed to add together several algebraic expressions, we aim at the same time to simplify the result by reducing it to as small a number of terms as possible by uniting several of the terms in one. This is done in articles 2 and 5, by reducing the quantity $x + x$ to $2x$, and the quantity $x + x + x$ to $3x$. It can take place only with respect to quantities expressed by the same letters, and which are for this reason called *similar* quantities. A literal quantity that is repeated any number of times is regarded as a unit, it is thus, that the quantities $2a$ and $3a$ considered as two and three units of a particular kind, form when added $5a$ or 5 units of the same kind. Also $4ab$ and $5ab$ make $9ab$.

In this case, the addition is performed with respect to the figures which precede the literal quantity, and which show how many times it is repeated. These figures are called *coefficients*. The coefficient then is the multiplier of the quantity before which it is placed, and it must be recollected, that when there is none expressed, unity is understood; for $1a$ is the same as a.

18. When it is proposed to unite any quantities whatever, as

$$4a + 5b \text{ and } 2c + 3d,$$

the sum total ought evidently to be composed of all the parts joined together; we must write then

$$4a + 5b + 2c + 3d.$$

If we have on the contrary

$$4a + 5b \quad \text{and} \quad 2c - 3d.$$

The sign $-$ must be retained in the sum, to mark as subtractive the quantity $3d$, which, as it is to be taken from $2c$, must necessarily diminish by so much the sum formed by uniting $2c$ with the first of the quantities proposed; we have then,

$$4a + 5b + 2c - 3d.$$

From these two examples it is evident, that *in algebra the addition of polynomials is performed by writing in order, one after the other, the quantities to be added with their proper signs, it being observed that the terms which have no signs before them are considered as having the sign $+$.*

The above operation is, properly speaking, only an indication by which the union of two compound quantities is made to consist in the addition and subtraction of a certain number of simple quantities; but, if the quantities to be added contained similar terms, these terms might be united by performing the operation upon their coefficients.

Let there be, for example, the quantities

$$\begin{aligned} &4a + 9b - 2c, \\ &2a - 3c + 4d, \\ &7b + c - e; \end{aligned}$$

the sum indicated would be, according to the rule just given,

$$4a + 9b - 2c + 2a - 3c + 4d + 7b + c - e.$$

But the terms $4a$, $+ 2a$, being formed of similar quantities, may be united in one sum equal to $6a$.

Also the terms $+ 9b$, $+ 7b$ give $+ 16b$.

The terms $- 2c$ and $- 3c$, being both subtractive, produce on the whole, the same effect as the subtraction of a quantity equal to their sum, that is to say, as the subtraction of $5c$; and as by virtue of the term $+ c$, we have another part c to be added, there will remain therefore to be subtracted only $4c$.

The sum of the expressions proposed then, will be reduced to

$$6a + 16b - 4c + 4d - e.$$

The last operation exhibited above, by which all similar terms are united in one, whatever signs they have, is called *reduction. It is performed by taking the sum of similar quantities having the sign $+$, that of similar quantities having the sign $-$, and subtracting the less of the two sums from the greater, and giving to the remainder the sign of the greater.*

It is to be remarked, that reduction is applicable to all algebraic operations.

The following examples of addition, with their answers, are intended as an exercise for the learner.

1. To add the quantities

$$\begin{array}{l} 7m + 3n - 14p + 17r \\ 3a + 9n - 11m + \ \ 2r \\ 5p - 4m + \ \ 8n \\ 11n - 2b - m - r + s. \end{array}$$

Answer, $7m+3n-14p+17r+3a+9n-11m+2r+5p-4m+8n$ $+11n-2b-m-r+s.$

By making the reduction, this quantity becomes

$$-9m + 31n - 9p + 18r + 3a - 2b + s,$$

or $$31n - 9m - 9p + 18r + 3a - 2b + s,$$

by beginning with the term having the sign $+$.

2. To add the quantities

$$\begin{array}{l} 11bc + 4ad - 8ac + 5cd \\ 8ac + 7bc - 2ad + 4mn \\ 2cd - 3ab + 5ac + an \\ 9an - 2bc - 2ad + 5cd. \end{array}$$

$$11bc + 4ad - 8ac + 5cd + 8ac + 7bc - 2ad + 4mn$$
$$2cd - 3ab + 5ac + an + 9an - 2bc - 2ad + 5cd.$$

By reducing this quantity it becomes

$$16bc + 5ac + 12cd + 4mn - 3ab + 10an.$$

Of the Subtraction of Algebraic Quantities.

20. The subtraction of single quantities, according to established usage, is represented by placing the sign $-$ between the quantity to be subtracted, and that from which it is to be taken; b subtracted from a is written $a - b$.

When the quantities are similar, the subtraction is performed directly by means of the coefficients.

If $3a$ be subtracted from $5a$, we have for a remainder $2a$.

With regard to the subtraction of polynomials, it is necssary to distinguish two cases.

1. If the terms of the quantity to be subtracted have each the sign $+$, we must clearly give to each the sign $-$, since it is required to deduct successively all the parts of the quantity to be subtracted.

If for example, from $5a - 9b + 2c$ we would take
$$2d + 3e + 4f,$$
we must write $5a - 9b + 2c - 2d - 3e - 4f.$

2. If any of the terms of the quantity to be subtracted have the sign $-$, we must give them the sign plus. Indeed, if from the quantity a we would take $b - c$, and should first write $a - b$, we should thus diminish a by the whole quantity b; but the subtraction ought to have been performed after having first diminished b by the quantity c; we have taken therefore this last quantity too much, and it is necessary to restore it with the sign $+$, which gives for the true result $a - b + c$.

This reasoning, which may be applied to all similar cases shows that the sign $-$ of c must be changed into the sign $+$; and by connecting this result with the preceding, we conclude, that *the subtraction of algebraic quantities is performed by writing them in order after the quantities, from which they are to be taken, having first changed the signs $+$ into $-$ and the signs $-$ into $+$.*

After this rule has been applied, the quantities are to be reduced when they will admit of it, according to the precept given in article 19, as may be seen in the following examples;

1. To subtract from $17a + 2m - 9b - 4c + 23d$
the quantity $51a - 27b + 11c - 4d.$

Result $17a + 2m - 9b - 4c + 23d$
$- 51a + 27b - 11c + 4d.$

When reduced it becomes
$$-34a + 2m + 18b - 15c + 27d,$$
or rather $2m - 34a + 18b - 15c + 27d.$

2. To subtract from $5ac - 8ab + 9bc - 4am$
the quantity $8am - 2ab + 11ac - 7cd.$

Result $5ac - 8ab + 9bc - 4am$
$- 8am + 2ab - 11ac + 7cd.$

Reduced it becomes
$$-6ac - 6ab + 9bc - 12am + 7cd,$$
or $9bc - 6ac - 6ab - 12am + 7cd.$

Of the Multiplication of Algebraic Quantities.

21. So far as letters are considered as expressing the numerical values of the quantities for which they stand, multiplication in algebra is to be regarded like multiplication in arithmetic.

(*Arith.* 21, 66.) Thus, *to multiply* a *by* b *is to compound with the quantity represented by* a *another quantity, in the same manner as the quantity represented by* b *is with unity.*

We have already explained, in articles 2 and 7, the signs used to indicate multiplication; and the product of a by b is expressed by $a \times b$, or by $a \cdot b$, or lastly, by ab.

We have often occasion to express several successive multiplications, as that of a by b, and that of the product ab by c, also that of this last product by d, and so on. In this case, it is evident, that the last result is a number having for *factors* the numbers a, b, c, d, (*Arith.* 22); and to give a general expression of this method, *we indicate the product by writing the factors composing it in order, one after the other, without any sign between them;* we have accordingly the expression $abcd$.

Reciprocally every expression, such as $abcd$ formed of several letters written in order one after the other, designates always the product of the numbers represented by these letters.

I have already availed myself of this method, in which the numerical coefficients are also included, since they are evidently factors of the quantity proposed. Indeed $15\,abcd$, designating the quantity $abcd$ taken 15 times, expresses likewise the product of the five factors 15, a, b, c, d.

It follows from this, that in order to indicate the multiplication of several simple quantities, such as $4abc$, $5def$, $3mn$, it is necessary to write the quantities in order, one after the other, without any sign between them, and it becomes

$$4abc5def3mn;$$

but since, as is shown in arithmetic, (art. 82) the order of the factors of a product may be changed at pleasure without altering the value of this product, we may avail ourselves of this principle, to bring together the numerical factors, the multiplication of which is performed by the rules of arithmetic; to express then this product, as indicated in the order $4 \cdot 5 \cdot 3\,abcdefmn$, we multiply together the numbers 4, 5, 3, which give simply

$$60\,abcdefmn.*$$

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* As the use of algebraic symbols abridges very much the demonstration of this proposition, I have thought it proper to suggest here a method by these symbols.

If we write the product $abcdef$ as follows, $abc \times de \times f$, and

23. The expression of the product may be much abridged when it contains equal factors. Instead of writing several times in order, the letter which represents one of the factors, it need be written only once with a number annexed, showing how many times it ought to have been written as a factor; but as this number indicates successive multiplications, it ought to be carefully distinguished from a coefficient, which indicates only additions. For this reason, it is placed on the right of the letter and a little higher up, while a coefficient is always placed on the left and on the same line.

Agreeably to this method, the product of a by a, which would be indicated according to article 21, by $a\,a$ becomes a^2. The 2 raised, denotes that the number, designated by the letter a, is twice a factor in the expression, to which it belongs. It ought not to be confounded with $2\,a$ which is only an abbreviation of $a + a$. To render evident the error, which would arise from mistaking one for the other, it is sufficient to substitute numbers instead of the letters. If we have, for example $a = 5$, $2\,a$ would become $2 \,.\, 5 = 10$, and $a^2 = a \times a = 5 \,.\, 5 = 25$.

Extending this method we should denote a product in which a is three times a factor by writing a^3 instead of $a\,a\,a$; also a^5 represents a product in which a is five times a factor, and is equivalent to $a\,a\,a\,a\,a$.

24. The products formed in this manner by the successive multiplications of a quantity, are called in general *powers* of that quantity.

The quantity itself, as a, is called the first power.

The quantity multiplied by itself, as $a\,a$, or a^2, is the second power. It is called also the *square*.

The quantity multiplied by itself twice in succession, as $a\,a\,a$, or a^3, is the third power, and is called also the *cube*.*

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change the order of the factors of the product to $e\,d$ instead of $d\,e$, (*Arith.* 27) it becomes $a\,b\,c \times e\,d \times f$, or $a\,b\,c\,e\,d\,f$. It is evident that we may, by analyzing the product differently, produce any change which we wish in the order of the factors of the product in question.

* The denominations *square* and *cube* refer to geometrical considerations. They interrupt the uniformity in the nomenclature of products formed by equal factors, and are very improper in algebra. But they are frequently used for the sake of conciseness.

In general, any power whatever is designated by the number of equal factors from which it is formed; a^5 or $a\,a\,a\,a\,a$ is the *fifth* power of a.

I take the number 3 to illustrate these denominations, and I have

1st. power	3
2d.	$3 \,.\, 3 = 9$
3d.	$3 \,.\, 3 \,.\, 3 = 9 \,.\, 3 = 27$
4th.	$3 \,.\, 3 \,.\, 3 \,.\, 3 = 27 \,.\, 3 = 81$
5th.	$3 \,.\, 3 \,.\, 3 \,.\, 3 \,.\, 3 = 81 \,.\, 3 = 243$
&c.	

The number which denotes the power of any quantity is called the *exponent* of this quantity.

When the exponent is equal to unity it is not written; thus a is the same as a^1.

It is evident then, that *to find the power of any number, it is necessary to multiply this number by itself as many times less one, as there are units in the exponent of the power.*

25. As the exponent denotes the number of equal factors, which form the expression of which it is a part, and as the product of two quantities must have each of these quantities as factors; it follows that the expression a^5 in which a is five times a factor, multiplied by a^3, in which a is three times a factor, ought to give a product in which a is eight times a factor, and consequently expressed by a^8, and that in general *the product of two powers of the same number ought to have for an exponent the sum of those of the multiplicand and multiplier.*

26. It follows from this, *that when two simple quantities have common letters, we may abridge the expression of the product of these quantities by adding together the exponents of such letters of the multiplicand and multiplier.*

For example, the expression of the product of the quantities $a^2\, b^3\, c$ and $a^4\, b^5\, c^2\, d$, which would be $a^2\, b^3\, c\, a^4\, b^5\, c^2\, d$, by the foregoing rule, art. 21, is abridged by collecting together the factors designated by the same letter, and

$$a^2\, a^4\, b^3\, b^5\, c\, c^2\, d,$$

becomes $\quad a^6\, b^8\, c^3\, d,$

by writing $\quad a^6$ instead of $a^2\, a^4$

b^8 instead of $b^3\, b^5$

c^3 instead of $c\, c^2$ or of $c^1\, c^2$.

27. As we distinguish powers by the number of equal factors from which they are formed, so also we denote any products by the number of simple factors or *firsts* which produce them; and I shall give to these expressions the name of *degrees*. The product $a^2 b^3 c$, for example, will be of the sixth degree, because it contains six simple factors, viz; 2 factors a, 3 factors b, and 1 factor c. It is evident that the factors a, b, and c, here regarded as firsts, are not so, except with respect to algebra, which does not permit us to decompose them; they may, notwithstanding, represent compound numbers, but we here speak of them only with respect to their general import.*

The coefficients expressed in numbers are not considered in estimating the degree of algebraic quantities; we have regard only to the letters.

It is evident, (21, 25) that when we multiply two simple quantities the one by the other, the number which marks the degree of the product is the sum of those which mark the degree of each of the simple quantities.

28. The multiplication of compound quantities consists in that of simple quantities, each term of the multiplicand and multiplier being considered by itself; as in arithmetic we perform the operation upon each figure of the numbers which we propose to multiply. (*Arith.* 33.) The particular products added together make up the whole product. But algebra presents a circumstance which is not found in numbers. These have no negative terms or parts to be subtracted, the units, tens, hundreds, &c. of which they consist, are always considered as added together, and it is very evident, that the whole product must be composed of the sum of the products of each part of the multiplicand by each part of the multiplier.

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* We apply the term *dimensions*, generally to what I have here called *degrees*, in conformity to the analogy already pointed out in the note to page 29. This example sufficiently proves the absurdity of the ancient nomenclature, borrowed from the circumstance, that the products of 2 and 3 factors, measure respectively the areas of the surfaces and the bulks of bodies, the former of which have two and the latter three dimensions; but beyond this limit the correspondence between the algebraic expressions and geometrical figures fails, as extension can have only three dimensions.

The same is true of literal expressions when all the terms are connected together by the sign $+$.

$$\begin{array}{ll} \text{The product of} & a + b \\ \text{multiplied by} & c \\ \hline \text{is} & ac + bc \end{array}$$

and is obtained by multiplying each part of the multiplicand by the multiplier, and adding together the two particular products $a c$ and $b c$. The operation is the same when the multiplicand contains more than two parts.

If the multiplier is composed of several terms, it is manifest that the product is made up of the sum of the products of the multiplicand by each term of the multiplier.

$$\begin{array}{ll} \text{The product of} & a + b \\ \text{multiplied by} & c + d \\ \hline \text{is} \left\{ \begin{array}{l} \\ \\ \end{array} \right. & \begin{array}{r} ac + bc \\ + ad + bd; \end{array} \end{array}$$

for by multiplying first $a + b$ by c, we obtain $a c + b c$, then by multiplying $a + b$ by the second term d of the multiplier, we have $a d + b d$, and the sum of the two results gives

$$ac + bc + ad + bd$$

for the whole.

29. When the multiplicand contains parts to be subtracted, the products of these parts by the multiplier must be taken from the others, or in other words, have the sign — prefixed to them. For example,

$$\begin{array}{ll} \text{the product of} & a - b \\ \text{multiplied by} & c \\ \hline \text{is} & ac - bc; \end{array}$$

for each time that we take the entire quantity a, which was to have been diminished by b before the multiplication, we take the quantity b too much; the product $a c$ therefore, in which the whole of a is taken as many times as is denoted by the number c, exceeds the product sought by the quantity b, taken as many times as is denoted by the number c, that is by the product $b c$; we ought then to subtract $b c$ from $a c$, which gives, as above,

$$a c - b c.$$

The same reasoning will apply to each of the parts of the multiplicand, that are to be subtracted, whatever may be the number, and whatever may be that of the terms of the multiplier, pro-

vided they all have the sign $+$. Recollecting that the terms which have no sign are considered as having the sign $+$, we see by the examples, that the terms of the multiplicand affected by the sign $+$ give a product affected by the sign $+$, while those which have the sign $-$ give one having the sign $-$. It follows from this, that *when the multiplier has the sign $+$, the product has the same sign as the corresponding part of the multiplicand.*

30. The contrary takes place when the multiplier contains parts to be subtracted; the products arising from these parts must be put down with a sign, contrary to that which they would have had by the above rule. This may be shown by the following example.

Let the multiplicand be $\quad a - b$

and the multiplier $\quad c - d$

the product will be $\left\{\begin{array}{r} ac - bc \\ -ad + bd; \end{array}\right.$

for the product of the multiplicand, by the first term of the multiplier, will be by the last example $ac - bc$; but by taking the whole of c for the multiplier instead of c diminished by d, we take the quantity $a - b$ so many times too much as is denoted by the number d; so that the product $ac - bc$ exceeds that sought by the product of $a - b$ by d. Now this last is, by what has been said, $ad - bd$, and in order to subtract it from the first it is necessary to change the signs (20). We have then

$ac - bc - ad + bd$ for the result required.

31. Agreeably to the above examples, we conclude, that *the multiplication of polynomials is performed by multiplying successively, according to the rules given for simple quantities* (21—26), *all the terms of the multiplicand by each term of the multiplier, and by observing that each particular product must have the same sign, as the corresponding part of the multiplicand, when the multiplier has the sign $+$, and the contrary sign when the individual multiplier has the sign $-$.*

If we develop the different cases of this last rule, we shall find,

1. That a term having the sign $+$, multiplied by a term having the sign $+$, gives a product having the sign $+$;

2. That a term having the sign $-$, multiplied by a term having the sign $+$, gives a product which has the sign $-$;

3. That a term having the sign $+$, multiplied by a term having the sign $-$, gives a product which has the sign $-$;

4. That a term having the sign —, multiplied by a term having the sign —, gives a product which has the sign +.

It is evident from this table, that *when the multiplicand and multiplier have the same sign, the product has the sign +, and that when they have different signs, the product has the sign —.*

To facilitate the practice of the multiplication of polynomials, I have subjoined a recapitulation of the rules to be observed.

1. *To determine the sign of each particular product according to the rule just given;* this is the rule for the signs.

2. *To form the coefficients by taking the product of those of each multiplicand and multiplier* (22); this is the rule for the coefficients.

3. *To write in order, one after the other, the different letters contained in each multiplicand and multiplier* (21); this is the rule for the letters.

4. *To give to the letters, common to the multiplicand and multiplier, an exponent equal to the sum of the exponents of these letters in the multiplicand and multiplier* (25); this is the rule for the exponents.

32. The example below will illustrate all these rules.

Multiplicand	$5a^4 - 2a^3b + 4a^2b^2$
Multiplier	$a^3 - 4a^2b + 2b^3$
Several products.	$5a^7 - 2a^6b + 4a^5b^2$ $-20a^6b + 8a^5b^2 - 16a^4b^3$ $+10a^4b^3 - 4a^3b^4 + 8a^2b^5$
Result reduced	$5a^7 - 22a^6b + 12a^5b^2 - 6a^4b^3 - 4a^3b^4 + 8a^2b^5$.

The first line of the several products contains those of all the terms of the multiplicand by the first term a^3 of the multiplier; this term being considered as having the sign +, the products which it gives have the same signs as the corresponding terms of the multiplicand (31).

The first term $5a^4$ of the multiplicand having the sign plus, we do not write that of the first term of the product, which would be +; the coefficient 5 of a^4 being multiplied by the coefficient 1 of a^3, gives 5 for the coefficient of this product; the sum of the two exponents of the letter a is 4 + 3, or 7, the first term of the product then is $5a^7$.

The second term $-2a^3b$ of the multiplicand having the sign —, the product has the sign minus; the coefficient 2 of a^3b mul-

tiplied by the coefficient 1 of a^3, gives 2 for the coefficient of the product; the exponent of the letter a, common to the two terms which we multiply, is $3 + 3$, or 6, and we write after it the letter b, which is found only in the multiplicand. The second term of the product then is $-2a^6b$.

The third term $+4a^2b^2$ gives a product affected with the sign $+$, and by the rules applied to the two preceding terms, we find it to be $+4a^5b^2$.

The second line contains the products of all the terms of the multiplicand by the second term $-4a^2b$ of the multiplier. This last having the sign $-$, all the products which it gives must have the signs contrary to those of the corresponding terms of the multiplicand; the coefficients, the letters, and the exponents are determined as in the preceding line.

The third line contains the products of all the terms of the multiplicand by the third term $+2b^3$ of the multiplier. This term having the sign $+$, all the products which it gives have the same sign as the corresponding terms of the multiplicand.

After having formed all the several products which compose the whole product, we examine carefully this last, to see whether it does not contain similar terms; if it does, we reduce them according to the rule (19), observing that two terms are similar, which consist of the same letters under the same exponents. In this example there are three reductions, viz;

$-2a^6b$ and $-20a^6b$, which give $-22a^6b$;
$+4a^5b^2$ and $+8a^5b^2$, which give $+12a^5b^2$;
$-16a^4b^3$ and $+10a^4b^3$, which give $-6a^4b^3$.

These reductions being made, we have for the result the last line of the example.

See another example to exercise the learner, which is easily performed after what has been said.

Multiplicand	$5a^4b^2+7a^3b^3-15a^5c+23b^2d^4-17bc^3d^2-9abcdm^2$
Multiplier	$11b^3-8c^3+5abc-2bdm$
Several products.	$55a^4b^5+77a^3b^6-165a^5b^3c+253b^5d^4-187b^4c^3d^2-99ab^4cdm^2$
	$-40a^4b\ c^3-56a^3b^3c^3+120a^5c^4-184b^2c^3d^4+136bc^6d^2+72abc^4dm^2$
	$+25a^5b^3c+35a^4b^4c-75a^6bc^2+115ab^3cd^4-85ab^2c^4d^2-45a^2b^2c^2dm^2$
	$-10a^4b^3dm-14a^3b^4dm+30a^5bcdm-46b^3d^5m+34b^2c^3d^3m+18ab^2cd^2m^3$
Result reduced.	$55a^4b^5+77a^3b^6-140a^5b^3c+253b^5a^4-187b^4c^3d^2-99ab^4cdm^2-40a^4b^2c^3-56a^3b^3c^3$
	$+120a^5c^4-184b^2c^3d^4+136bc^6d^2+72abc^4dm^2+35a^4b^4c-75a^6bc^2+115ab^3cd^4-85ab^2c^4d^2$
	$-45a^2b^2c^2dm^2-10a^4b^3dm-14a^3b^4dm+30a^5bcdm-46b^3d^5m+34b^2c^3d^3m+18ab^2cd^2m^3.$

33. From the manner of proceeding in multiplication, it is evident that if all the terms of the multiplicand are of the same degree (27), and those of the multiplier are also of the same degree, all the terms of the product will be of a degree denoted by the sum of the numbers, which mark the degree of the terms of each of the factors.

In the first example, the multiplicand is of the fourth degree, the multiplier of the third; and the product is of the seventh.

In the second example, the multiplicand is of the sixth degree, the multiplier of the third; and the product is of the ninth.

Expressions of the kind just referred to, the terms of which are all of the same degree, are called *homogeneous* expressions. The above remark, with respect to their products, may serve to prevent occasional errors, which one may commit by forgetting some of the factors in the several parts of the multiplication.

34. Algebraic operations performed upon literal quantities, as they permit us to see how the several parts of the quantities concur to form the results, often make known some general properties of numbers independent of every system of notation. The multiplications that follow, lead to conclusions of the greatest importance, and of frequent use in the subsequent parts of this work.

$$\begin{array}{r} a+b \\ a-b \\ \hline a^2+ab \\ -ab-b^2 \\ \hline a^2-b^2 \end{array} \qquad \begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \\ +ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

$$\begin{array}{r} a^2+2ab+b^2 \\ a+b \\ \hline a^3+2a^2b+ab^2 \\ +a^2b+2ab^2+b^3 \\ \hline a^3+3a^2b+3ab^2+b^3. \end{array}$$

It appears from the first of these products, that the quantity $a+b$, multiplied by $a-b$, gives a^2-b^2; whence it is evident that, *if we multiply the sum of two numbers by their difference, the product will be the difference of the squares of these numbers.*

If we take, for example, the sum 11 of the numbers 7 and 4,

and multiply it by the difference 3 of these numbers, the product 3×11, or 33, will be equal to the difference between 49, the square of 7, and 16, the square of 4.

By the second example, in which $a + b$ is twice a factor, we learn; *that the second power, or the square of a quantity composed of two parts* a *and* b *contains the square of the first part, plus double the product of the first part by the second, plus the square of the second.*

The third example, in which we have multiplied the second power of $a + b$ by the first, shows; that, *the third power or cube of a quantity composed of two parts contains the cube of the first, plus three times the square of the first multiplied by the second, plus three times the first multiplied by the square of the second plus the cube of the second.*

35. As we have often occasion to decompose a quantity into its factors, and as the algebraic operations are dispensed with, when it can be done, in order to exhibit the formation of the quantities to be considered, as distinctly as possible, it is necessary to fix upon some signs proper to indicate multiplication between complex quantities.

We use indeed the marks of a parenthesis to comprehend the factors of a product. The expression

$$(5a^4 - 3a^2b^2 + b^4)(4ab^2 - ac^2 + d^3)(b^2 - c^2),$$

for example, indicates the product of the compound quantities

$$5a^4 - 3a^2b^2 + b^4,\ 4ab^2 - ac^2 + d^3, \text{ and } b^2 - c^2.$$

Bars were used formerly by some authors placed over the factors thus,

$$\overline{5a^4 - 3a^2b^2 + b^4} \times \overline{4ab^2 - ac^2 + d^3} \times \overline{b^2 - c^2};$$

but as these may happen to be too long or too short, they are liable to more uncertainty than the marks of a parenthesis, which can never admit of any doubt with respect to the quantity belonging to each factor. They have accordingly been preferred.

Of the Division of Algebraic Quantities.

36. Algebraic division, like division in arithmetic, is to be regarded as an operation designed to discover one of the factors of a given product, when the other is known. According to this definition, the quotient multiplied by the divisor must produce anew the dividend.

By applying what is here said to simple quantities we shall see by art. 21, that the dividend is formed from the factors of the divisor and those of the quotient; whence, *by suppressing in the dividend all the factors which compose the divisor, the result will be the quotient sought.*

Let there be, for example, the simple quantity $72a^5 b^3 c^2 d$ to be divided by the simple quantity $9a^3 b c^2$; according to the rule above given, we must suppress in the first of these quantities the factors of the second, which are respectively

$$9,\ a^3,\ b, \text{ and } c^2.$$

It is necessary then, in order that the division may be performed, that these factors should be in the dividend. Taking them in order, we see in the first place that the coefficient 9 of the divisor, ought to be a factor of the coefficient 72 of the dividend, or that 9 ought to divide 72 without a remainder. This is in fact the case, since $72 = 9 \times 8$. By suppressing then the factor 9, there will remain the factor 8 for the coefficient of the quotient.

It follows moreover, from the rules of multiplication (25), that the exponent 5 of the letter a in the dividend, is the sum of the exponents belonging to the divisors and quotient; this last exponent therefore will be the difference between the two others, or $5 - 3 = 2$. Thus the letter a has in the quotient the exponent 2. For the same reason, the letter b has in the quotient an exponent equal to $3 - 1$, or 2. The factor c^2 being common to the dividend and divisor is to be suppressed, and we have

$$8a^2 b^2 d$$

for the quotient required.

The same will apply to every other case; we conclude then, that, *in order to effect the division of simple quantities, the course to be pursued is,*

To divide the coefficient of the dividend by that of the divisor;

To suppress in the dividend the letters which are common to it and the divisor, when they have the same exponent; and when the exponent is not the same, to subtract the exponent of the divisor from that of the dividend, the remainder being the exponent to be affixed to the letter in the quotient;

To write in the quotient the letters of the dividend which are not in the divisor.

37. If we apply the rule now given for obtaining the exponent of the letters of the quotient, to a letter which has the same

exponent in the dividend and divisor, we shall find zero to be the exponent which it ought to have in the quotient; a^3 divided by a^3, for example, gives a^0. To understand what is the import of such an expression, it is necessary to go back to its origin and to consider, that if we represent the quotient arising from the division of a quantity by itself, it ought to answer to unity, which expresses how many times any quantity is contained in itself. It follows from this, that *the expression* a^0 *is a symbol equivalent to unity, and may consequently be represented by* 1. We may then omit writing the letters which have zero for their exponent, since each of them signifies nothing but unity. Thus $a^3\,b\,c^2$ divided by $a^2\,b\,c^2$, gives $a^1\,b^0\,c^0$, which becomes a, as is very evident by suppressing the common factors of the dividend and divisor.

We see by this, that the proposition, *every quantity which has zero for its exponent, is equal to* 1, is nothing, properly speaking, but the explanation of a conclusion to which we are brought by the common manner of writing the powers of quantities by exponents.

In order that the division may be performed, it is necessary, 1. that the divisor should have no letter which is not found in the dividend; 2. that the exponent of any letter in the divisor should not exceed that of the same letter in the dividend; 3. that the coefficient of the divisor should exactly divide that of the dividend.

38. When these conditions do not exist, the division can only be indicated in the manner pointed out in the 2d article. Still we should endeavour to simplify the fraction by suppressing such factors, as are common to the dividend and divisor, if there are any such; for (*Arith.* 57) it is manifest, that the theory of arithmetical fractions rests upon principles which are independent of every particular value of their terms, and which would apply to fractions represented by letters, as well as to those which are represented by numbers.

According to these principles, *we in the first place suppress the numerical factors common to the dividend and divisor, and then the letters which are common to the dividend and divisor, and which have the same exponent in each. When the exponent is not the same in each, we subtract the less from the greater, and affix the remainder, as the exponent to the letter, which is written only in that term of the fraction which has the highest exponent.*

The following example will illustrate this rule.

Let $48\, a^3\, b^5\, c^2\, d$ be divided by $64\, a^3\, b^3\, c^4\, e$; the quotient can only be indicated in the form of a fraction

$$\frac{48\, a^3\, b^5\, c^2\, d}{64\, a^3\, b^3\, c^4\, e}.$$

But the coefficients 48 and 64 being divisible by 16, by suppressing this common factor, the coefficient of the numerator becomes 3, and that of the denominator 4. The letter a having the same exponent 3 in the two terms of the fraction, it follows that a^3 is a factor common to the dividend and divisor, and may consequently be suppressed.

To find the number of factors b common to the two terms of the fraction, we must divide the higher b^5 by the lower b^3, according to the rule above given, and the quotient b^2 shows, that $b^5 = b^3 \times b^2$. Suppressing then the common factor b^3, there will remain in the numerator the factor b^2.

With respect to the letter c, the higher factor being c^4 of the denominator, if we divide it by c^2 we shall decompose it into $c^2 \times c^2$; and by suppressing the factor c^2 common to the two terms, this letter disappears from the numerator, but will remain in the denominator with the exponent 2.

Finally, the letters d and e will remain in their respective places, since in the state in which they are, they indicate no factor common to both.

By these several operations the proposed fraction is reduced to

$$\frac{3\, b^2\, d}{4\, c^2\, e};$$

and it is the most simple expression of the quotient, except we give numerical values to the letters; in which case it might be further reduced by cancelling the common factors as before.

39. It ought to be remarked, that, if all the factors of the dividend enter into the divisor, which besides contains others peculiar to it, it is necessary after suppressing the former to put unity in the place of the dividend, as the numerator of the fraction. In this case indeed we may suppress all the terms of the numerator, or, in other words, divide the two terms of the fraction by the numerator; but this being divided by itself must give unity for the quotient, which becomes the new numerator.

Suppose for example the fraction

$$\frac{4\, a^2\, b\, c}{12\, a^2\, b^3\, c\, d};$$

the factors 12, a^2, b^3, and c may be divided respectively by the factors 4, a^2, b, and c, or we may divide the two terms of the fraction by the numerator $4\,a^2\,b\,c$. Now the quantity $4\,a^2\,b\,c$, divided by itself, gives 1 for the quotient, and the quantity $12\,a^2\,b^3\,c\,d$, divided by the first, gives by the above rules $3\,b^2\,d$; the new fraction then is

$$\frac{1}{3\,b^2\,d}.$$

40. It follows from the rules of multiplication, that when a compound quantity is multiplied by a simple quantity, this last becomes a factor common to all the terms of the former. We may make use of this observation to simplify fractions of which the numerator and denominator are polynomials, having factors that are common to all their terms.

Let there be the expression

$$\frac{6\,a^4 - 3\,a^2\,b\,c + 12\,a^2 c^2}{9\,a^2\,b - 15\,a^2\,c + 24\,a^3};$$

by examining the quantity $6\,a^4 - 3\,a^2\,bc + 12\,a^2\,c^2$, we see that the factor a^2 is common to all the terms, since $a^4 = a^2 \times a^2$, and that, besides, 6, 3, and 12 are divisible by 3; so that,

$6a^4 - 3a^2bc + 12a^2c^2 = 2a^2 \times 3a^2 - bc \times 3a^2 + 4c^2 \times 3a^2$.

Also the denominator has for a common factor $3\,a^2$; for the factors a^2 and 3 enter into all the terms, and we have

$9\,a^2b - 15\,a^2c + 24\,a^3 = 3b \times 3\,a^2 - 5\,c \times 3\,a^2 + 8\,a \times 3a^2$.

Suppressing therefore the $3\,a^2$ as often in the numerator as in the denominator, the proposed fraction will become

$$\frac{2\,a^2 - b\,c + 4c^2}{3\,b - 5c + 8\,a}.$$

41. I pass now to the case where the numerator and denominator are both compound, and in which one cannot perceive at first whether the divisor is or is not a factor of the dividend.

As the divisor multiplied by the quotient must produce the dividend, it is necessary that this last should contain all the several products of each term of the divisor by each term of the quotient; and, if we could find the products arising from each particular term of the divisor, by dividing them by this term, which is known, we should obtain those of the quotient, after the same manner as in arithmetic we discover all the figures of the quotient by dividing successively by the divisor the numbers, which we regard as the several products of this divisor by the

different figures of the quotient. But in numbers the several products present themselves in order, beginning with the units at the last place on the left, on account of the subordination established between the units of each figure of the dividend according to the rank which they hold. But as this is not the case in algebra, we supply the want of such an arrangement by disposing all the terms of the dividend and divisor in the order of the exponents of the power of the same letter, beginning with the highest and proceeding from left to right, as may be seen with reference to the letter a in the quantities

$$5a^7 - 22a^6b + 12a^5b^2 - 6a^4b^3 - 4a^3b^4 + 8a^2b^5,$$
$$5a^4 - 2a^3b + 4a^2b^2,$$

of which one is the product and the other the multiplicand in the example of art. 32. This is called *arranging* the proposed quantities.

When they are thus disposed, it is evident, that whatever be the factor by which it is necessary to multiply the second to obtain the first, the term $5a^7$, with which this begins, results from the multiplication of $5a^4$, with which the other begins, by the term in the factor sought, in which a has the highest exponent, and which takes the first place in this factor when the terms of it are arranged with reference to the letter a. By dividing then the simple quantity $5a^7$ by the simple quantity $5a^4$, the quotient a^3 will be the first term of the factor sought. Now as the entire product ought by the rules of multiplication to contain the several particular products arising from the multiplication of the whole multiplicand by each term of the multiplier, it follows that the quantity here taken for the dividend, ought to contain the products of all the terms of the divisor, $5a^4 - 2a^3b + 4a^2b^2$, by the first term of the quotient a^3; and consequently, if we subtract from the dividend these products, which are $5a^7 - 2a^6b + 4a^5b^2$, the remainder $-20a^6b + 8a^5b^2 - 6a^4b^3 - 4a^3b^4 + 8a^2b^5$ will contain only those, which result from the multiplication of the divisor by the second, third, &c. terms of the quotient.

The remainder then may be considered as a part of the dividend, and its first term, in which a has the highest exponent, cannot be obtained, otherwise than by the multiplication of the first term of the divisor by the second term of the quotient. But the first term of this part of the dividend having the sign $-$, it is necessary to assign that which is to be prefixed to the corresponding term of the quotient. This is easily done by the first

rule art. 31, for the quantity $- 20a^6b$, being regarded as a part of the product, having a sign contrary to that of the multiplicand $5a^4$, it follows that the multiplier must have the sign $-$. Division then being performed upon the simple quantities, $-20a^6b$ and $5a^4$, gives $-4a^2b$ for the second term of the quotient. If now we multiply this by all the terms of the divisor, and subtract the product from the partial dividend, the remainder $+10a^4b^3 - 4a^3b^4 + 8a^2b^5$ will contain only the products of the third &c. terms of the quotient.

Regarding this remainder as a new dividend, its first term $10a^4b^3$ must be the product of the first term of the divisor by the third of the quotient, and consequently this last is obtained by dividing the simple quantities, $10a^4b^3$ and $5a^4$, the one by the other. The quotient $2b^3$ being multiplied by the whole of the divisor furnishes products, the subtraction of which, exhausting the remaining dividend, proves that the quotient has only three terms.

If the question had been such as to require a greater number of terms, they might evidently have been found like the preceding, and if, as we have supposed, the dividend has the divisor for a factor, the subtraction of the product of this divisor by the last term of the quotient ought always to exhaust the corresponding dividend.

42. To facilitate the practice of the above rules;

1. *We dispose the dividend and divisor, as for the division of numbers, by arranging them with reference to some letter, that is, by writing the terms in the order of the exponents of this letter, beginning with the highest;*

2. *We divide the first term of the dividend by the first term of the divisor, and write the result in the place of the quotient;*

3. *We multiply the whole divisor by the term of the quotient just found, subtract it from the dividend, and reduce similar terms.*

4. *We regard this remainder as a new dividend, the first term of which we divide by the first term of the divisor, and write the result as the second term of the quotient, and continue the operation till all the terms of the dividend are exhausted.*

Recollecting that when a product has the same sign as the multiplicand, the multiplier has the sign $+$, and, that when a product has the contrary sign to that of the multiplicand, the multiplier has the sign $-$ (31), we infer that, *when the term of the dividend and the first term of the divisor have the same sign, the*

quotient ought to have the sign $+$, *and, if they have contrary signs, the quotient ought to have the sign* $-$; this is the rule for the signs.

The individual parts of the operation are performed by the rule for the division of simple quantities.

We divide the coefficient of the dividend by that of the divisor; this is the rule for the coefficients.

We write in the quotient the letters common to the dividend and divisor with an exponent equal to the difference of the exponents of these letters in the two terms, and the letters which belong only to the dividend; these are the rules for the letters and exponents.

43. To apply these rules to the quantities,

$$5\,a^7 - 22\,a^6\,b + 12\,a^5\,b^2 - 6\,a^4\,b^3 - 4\,a^3\,b^4 + 8\,a^2\,b^5,$$
$$5\,a^4 - 2\,a^3\,b + 4\,a^2\,b^2,$$

which have been employed as an example above, we place them as we place the dividend and divisor in arithmetic.

Dividend.	*Divisor.*
$5a^7-22a^6b+12a^5b^2-6a^4b^3-4a^3b^4+8a^2b^5$	$5a^4-2a^3b+4a^2b^2$
$-5a^7+2a^6b-4a^5b^2$	*Quotient.*
	$a^3-4a^2b+2b^3$
Rem. $-20a^6b+8a^5b^2-6a^4b^3-4a^3b^4+8a^2b^5$	
$+20a^6b-8a^5b^2+16a^4b^3$	
rem. $+10a^4b^3-4a^3b^4+8a^2b^5$	
$-10a^4b^3+4a^3b^4-8a^2b^5$	
0.	

The sign of the first term $5\,a^7$ of the dividend being the same as that of $5\,a^4$, the first term of the divisor, the sign of the quotient must be $+$, but, as it is the first term, the sign is omitted.

By dividing $5\,a^7$ by $5\,a^4$, we have for the quotient a^3, which we write under the divisor.

Multiplying successively the three terms of the divisor by the first term a^3 of the quotient, and writing the products under the corresponding terms of the dividend, the signs being changed to denote their subtraction (20), we have the quantity

$$-5\,a^7 + 2\,a^6\,b - 4\,a^5\,b^2,$$

which with the dividend being reduced, we obtain for a remainder

$$-20\,a^6\,b + 8\,a^5\,b^2 - 6\,a^4\,b^3 - 4\,a^3\,b^4 + 8\,a^2\,b^5.$$

By continuing the division with this remainder, the first term $-20\,a^6\,b$, divided by $5\,a^4$, will give for a quotient $4\,a^2\,b$, this quotient having the sign $-$, as the dividend and divisor have

different signs. Multiplying it by all the terms of the divisor and changing the signs, we obtain the quantity

$$20\,a^6\,b - 8\,a^5\,b^2 + 16\,a^4\,b^3,$$

which taken with the dividend and reduced, gives for a remainder

$$+ 10\,a^4\,b^3 - 4\,a^3\,b^4 + 8\,a^2\,b^5.$$

Dividing the first term of this new dividend, $10\,a^4\,b^3$, by the first term, $5\,a^4$, of the divisor, and multiplying the whole divisor by the result $+ 2\,b^3$, writing the products under the dividend, the signs being changed, and making the reduction, we find that nothing remains, which shows that $+ 2\,b^3$ is the last term of the quotient sought. The quotient therefore has for its expression $a^3 - 4\,a^2\,b + 2\,b^3$.

44. It is proper to remark here, that in division, the multiplication of the different terms of the quotient by the divisor often produces terms that are not to be found in the dividend, and which it is necessary to divide by the first term of the divisor. These terms are such as destroy themselves, since the dividend has been formed by the multiplication of the two factors, the quotient and the divisor. See a remarkable example of these reductions;

Let $a^3 - b^3$ be divided by $a - b$.

Division.	*Multiplication.*
$a^3 - b^3 \quad \vert\; a - b$	$a - b$
$- a^3 + a^2\,b \quad \vert\; a^2 + a\,b + b^2$	$a^2 + a\,b + b^2$
$+ a^2\,b - b^3$	$a^3 - a^2\,b$
$- a^2\,b + a\,b^2$	$+ a^2\,b - a\,b^2$
$+ a\,b^2 - b^3$	$+ a\,b^2 - b^3$
$- a\,b^2 + b^3$	Result $a^3 - b^3$.
$0 \quad 0$	

The first term a^3 of the dividend, divided by the first term a of the divisor, gives for the quotient a^2; multiplying this quotient by the divisor, and changing the signs of the products, we have $- a^3 + a^2\,b$; the first term $- a^3$ destroys the first term of the dividend, but there remains the term $a^2\,b$, which is not found at first in the dividend. As it contains the letter a, we can divide it by the first term of the divisor, and obtain $+ a\,b$. Multiplying this quotient by the divisor, and changing the signs of the products, we have $- a^2\,b + a\,b^2$; the term $- a^2\,b$ cancels the one above it, but there remains the term $+ a\,b^2$, which is not in the

dividend. This being divided by a gives for the quotient $+ b^2$; multiplying this quotient by the divisor and changing the signs, we have $-ab^2 + b^3$; the first term $-ab^2$ destroys the first term of the dividend, and the second $+ b^3$ destroys the other $-b^3$.

The mechanical part of the operation will be better understood, if we look for a moment at the multiplication of the quotient $a^2 + ab + b^2$ by the divisor $a - b$. We see that all the terms reproduced in the process of dividing are those which destroy each other in the result of the multiplication.

45. It sometimes happens that the quantity, with reference to which the arrangement is made, has the same power in several terms both of the dividend and divisor. In this case, the terms should be written in the same column, one under the other, the remaining ones being disposed with reference to another letter.

Let there be

$$-a^4 b^2 + b^2 c^4 - a^2 c^4 - a^6 + 2a^4 c^2 + b^6 + 2b^4 c^2 + a^2 b^4,$$

to be divided by $a^2 - b^2 - c^2$.

Arranging the first of these quantities with reference to the letter a, we place in the same column the terms $-a^4 b^2$ and $+2a^4 c^2$, in another, the terms $+a^2 b^4$ and $-a^2 c^4$; and in the last column, the three terms $+b^6$, $+2b^4 c^2$, $+b^2 c^4$, disposing them with reference to the letter b, as may be seen in the next page.

The first term a^6 of the dividend being divided by the first term a^2 of the divisor, gives for the first term of the quotient $-a^4$; forming the products of this quotient by all the terms of the divisor, changing the signs of the products in order to subtract them from the dividend, and placing in the same column the terms containing the same power of a, we have, after the reduction of similar terms, the first remainder, which we take for the second dividend.

The first term $-2a^4 b^2$ of this new dividend, being divided by a^2, gives for the second term of the quotient $-2a^2 b^2$; forming the products of this quotient by all the terms of the divisor, changing the signs of the products to indicate their subtraction from the dividend, and placing in the same column the terms containing the same power of a, we have, after the reduction of similar terms, the second remainder, which we take for the third dividend.

The operation being continued in the same manner with the

second remainder and the following ones, we shall have three terms in the quotient. The last being multiplied by all the terms of the divisor, furnishes products which, being subtracted from the fourth remainder, exhaust it entirely. As the division admits of being exactly performed, it follows, that the divisor is a factor of the dividend.

$$
\begin{array}{lllll|l}
 & -a^6 - & a^4 b^2 + a^2 b^4 + b^6 & & & a^2 - b^2 - c^2 \\
 & & +2a^4c^2 - a^2c^4 + 2b^4c^2 & & & \overline{-a^4 - 2a^2b^2 - b^4} \\
 & & \qquad\qquad\qquad + b^2c^4 & & & \quad + a^2c^2 - b^2c^2 \\
 & +a^6 - a^4b^2 & & & & \\
 & \quad\; - a^4c^2 & & & & \\
\hline
\text{1st. rem.} & -2a^4b^2 + a^2b^4 + b^6 & & & & \\
 & + a^4c^2 - a^2c^4 + 2b^4c^2 & & & & \\
 & \qquad\qquad\qquad + b^2c^4 & & & & \\
 & +2a^4b^2 - 2a^2b^4 & & & & \\
 & \qquad\quad - 2a^2b^2c^2 & & & & \\
\hline
\text{2d rem.} & + a^4c^2 - a^2b^4 + b^6 & & & & \\
 & \qquad\quad -2a^2b^2c^2 + 2b^4c^2 & & & & \\
 & \qquad\quad - a^2c^4 + b^2c^4 & & & & \\
 & - a^4c^2 + a^2b^2c^2 & & & & \\
 & \qquad\quad + a^2c^4 & & & & \\
\hline
\text{3d rem.} & \qquad\quad - a^2b^4 + b^6 & & & & \\
 & \qquad\quad - a^2b^2c^2 + 2b^4c^2 & & & & \\
 & \qquad\qquad\qquad + b^2c^4 & & & & \\
 & \qquad\quad + a^2b^4 - b^6 & & & & \\
 & \qquad\qquad\qquad - b^4c^2 & & & & \\
\hline
\text{4th rem.} & \qquad\quad - a^2b^2c^2 + b^4c^2 & & & & \\
 & \qquad\qquad\qquad + b^2c^4 & & & & \\
 & \qquad\quad + a^2b^2c^2 - b^4c^2 & & & & \\
 & \qquad\qquad\qquad - b^2c^4 & & & & \\
\hline
 & \qquad\qquad\quad 0 \qquad 0 & & & &
\end{array}
$$

46. The form under which a quantity appears, will sometimes immediately suggest the factors into which it may be decomposed. If we have, for example,

$$8a^6 - 4a^3b^2 + 4a^3 + 2a^3 - b^2 + 1,$$

to be divided by $2a^3 - b^2 + 1$; as the divisor forms the three last terms of the dividend, it is only necessary to see if it is a factor of the three first; but these have obviously for a common factor $4a^3$, for $8a^6 - 4a^3b^2 + 4a^3 = 4a^3(2a^3 - b^2 + 1)$.

The dividend then may be represented by

$$4a^3(2a^3 - b^2 + 1) + 2a^3 - b^2 + 1,$$

or

$$(2a^3 - b^2 + 1)(4a^3 + 1).$$

The division is performed at once by suppressing the factor $2a^3 - b^2 + 1$, equal to the divisor, and the quotient will be $4a^3 + 1$.

After a little practice, methods of this kind will readily occur, by which algebraic operations are abridged.

By frequent exercise in examples of this kind, the resolution of a quantity into its factors is at length easily performed; and it is often rendered very conspicuous, when, instead of performing the operations represented, they are only indicated.

Of Algebraic Fractions.

47. When we apply the rules of algebraic division to quantities, of which the one is not a factor of the other, we perceive the impossibility of performing it, since in the course of the operation we arrive at a remainder, the first term of which is not divisible by that of the divisor. See an example;

$$\begin{array}{r|l} a^3 + a^2 b + 2b^3 & a^2 + b^2 \\ -a^3 - ab^2 & \overline{a + b} \\ \hline \text{1st rem.} \quad a^2 b - ab^2 + 2b^3 & \\ -a^2 b - b^3 & \\ \hline -ab^2 + b^3. & \end{array}$$

The first term, $-ab^2$, of the second remainder cannot be divided by a^2, the first term of the divisor; so that the process is arrested at this point. We can however, as in arithmetic, annex to the quotient $a + b$ the fraction $\frac{-ab^2 + b^3}{a^2 + b^2}$, having the remainder for the numerator, and the divisor for the denominator; and the quotient will be

$$a + b + \frac{b^3 - ab^2}{a^2 + b^2}.$$

It is evident, that *the division must cease, when we come to a remainder, the first term of which does not contain the letter with reference to which the terms are arranged, or to a power inferior to that of the same letter in the first term of the divisor.*

48. When the algebraic division of the two quantities cannot be performed, the expression of the quotient remains indicated under the form of a fraction, having the dividend for the nume-

rator, and the divisor for the denominator; and to abridge it as much as possible, we should see if the dividend and divisor have not common factors, which may be cancelled (38). But when the terms of the fraction are polynomials, the common factors are not so easily found, as when they are simple quantities. They are in general to be sought by a method analogous to that, which is given in arithmetic for finding the *greatest common divisor* of two numbers.

We cannot assign the relative magnitudes of algebraic expressions, as we do not give values to the letters which they contain; the denomination of *greatest common divisor* therefore, applied to these expressions, ought not to be taken altogether in the same sense as in arithmetic.

In algebra, we are to understand by the *greatest common divisor* of two expressions, that which contains the most factors in all its terms, or which is of the highest degree (27). Its determination rests, as in arithmetic, upon this principle; *Every common divisor to two quantities must divide the remainder after their division.*

The demonstration given in arithmetic (art. 61) is rendered clearer by employing algebraic symbols. If we represent the common divisor by D, the two quantities proposed might be expressed by the products AD and BD, formed from the common divisor and the factor by which it is multiplied in each of the quantities. This being supposed, if Q stands for the entire quotient, and R for the remainder resulting from the division of AD by BD, we have $AD = BD \times Q + R$ (*Arith.* 61); dividing now the two members of the equation by D, we obtain

$$A = BQ + \frac{R}{D};$$

and since the first member, which in this case must be composed of the same terms, as the second, is entire, it must follow, that $\frac{R}{D}$ is reduced to an expression without a divisor, that is to say, that R is divisible by D.

According to this principle, *we begin, as in arithmetic, by inquiring whether one of the quantities is not itself the divisor of the other; if the division cannot be exactly performed, we divide the first divisor by the remainder, and so on; and that remainder, which will exactly divide the preceding, will be the greatest common divisor of the two quantities proposed.* But it will be necessary, in

the divisions indicated, to have regard to what belongs to the nature of algebraic quantities.

We are not, in the first place, to seek a common divisor of two algebraic quantities, except when they have common letters; and we must select from them a letter, with reference to which the proposed expressions are to be arranged, and that is to be taken for the dividend in which this letter has the highest exponent, the other being the divisor.

Let there be the two quantities

$$3a^3 - 3a^2b + ab^2 - b^3,$$
$$4a^2b - 5ab^2 + b^3,$$

which are already arranged with reference to the letter a; we take the first for the dividend, and the second for the divisor. A difficulty immediately presents itself, which we do not meet with in numbers, and this is, that the first term of the divisor will not exactly divide the first term of the dividend, on account of the factors 4 and b in the one, which are not in the other. But the letter b being common to all the terms of the divisor and not to those of the dividend, it follows (40) that b is a factor of the divisor, and that it is not of the dividend. Now every divisor common to two quantities, can consist only of factors which are common to the one and to the other; if then there be such a divisor with respect to the two quantities proposed, it is to be looked for among the factors of the quantity $4a^2 - 5ab + b^2$, which remains of the quantity $4a^2b - 5ab^2 + b^3$, after suppressing b; so that the question reduces itself to finding the greatest common divisor of the two quantities

$$3a^3 - 3a^2b + ab^2 - b^3,$$
$$4a^2 - 5ab + b^2.$$

For the same reason that we may cancel in one of the proposed quantities the factor b which is not in the other, we may likewise introduce into this a new factor, provided it is not a factor of the first. By this step, the greatest common divisor, which can consist only of terms common to both, will not be affected. Availing myself of this principle, I multiply the quantity $3a^3 - 3a^2b + ab^2 - b^3$ by 4, which is not a factor of the quantity $4a^2 - 5ab + b^2$, in order to render the first term of the one divisible by the first term of the other.

I shall thus have for the dividend, the quantity

$$12a^3 - 12a^2b + 4ab^2 - 4b^3,$$

for the divisor the quantity

$$4a^2 - 5ab + b^2,$$

and the quotient will be $3a$.

Multiplying the divisor by this quotient, and subtracting the product from the dividend, I have for a remainder

$$3a^2b + ab^2 - 4b^3;$$

a quantity which, according to the principle stated at the commencement of this article, must have with $4a^2 - 5ab + b^2$, the same greatest common divisor as the first.

Profiting by the remarks made above, I suppress the factor b, common to all the terms of this remainder, and multiply it by 4, in order to render the first term divisible by that of the divisor; I have then for a dividend, the quantity

$$12a^2 + 4ab - 16b^2,$$

and for a divisor, the quantity

$$4a^2 - 5ab + b^2;$$

and the quotient thence arising is 3.

Multiplying the divisor by the quotient, and subtracting the product from the dividend, we obtain the remainder

$$19ab - 19b^2,$$

and the question is reduced to finding the greatest common divisor to this quantity, and

$$4a^2 - 5ab + b^2.$$

But the letter a, with reference to which the division is made, not being in the remainder, except of the first degree, while it is of the second degree in the divisor, it is this which must be taken for the dividend, and the remainder must be made the divisor.

Before beginning this new division, I expunge from the divisor $19ab - 19b^2$, the factor $19b$, common to both the terms, and which is not a factor of the dividend; I have then for a dividend, the quantity

$$4a^2 - 5ab + b^2,$$

and for a divisor

$$a - b.$$

The division leaves no remainder; so that $a - b$ is the greatest common divisor required.

By retracing these steps, we may prove *à posteriori*, that the quantity $a - b$ must exactly divide the two quantities proposed, and that it is the most compounded of those which will do it. In dividing by $a - b$ the two quantities proposed,

$$3a^3 - 3a^2b + ab^2 - b^3, \quad 4a^2b - 5ab^2 + b^3,$$

we resolve them as follows;

$$(3a^2 + b^2)(a - b), \quad (4ab - b^2)(a - b).$$

49. When the quantity, which we take for a divisor, contains several terms having the letter, with reference to which the arrangement is made, of the same degree, there are precautions to be used, without which the operation would not terminate. See an example of this.

Let there be the quantities

$$a^2 b + a c^2 - d^3, \quad ab - ac + d^2;$$

if we make the preparation as for common division,

$$\begin{array}{r|l} a^2 b + a c^2 - d^3 & ab - ac + d^2 \\ \cline{2-2} - a^2 b + a^2 c - a d^3 & a \\ \hline \text{Rem. } a^2 c + a c^3 - a d^2 - d^3, & \end{array}$$

by dividing, first, $a^2 b$ by ab, we have for the quotient a; multiplying the divisor by this quotient, and subtracting the products from the dividend, the remainder will contain a new term, in which a will be of the second degree, namely, $a^2 c$, arising from the product of $-ac$ by a. Thus no progress has been made; for by taking the remainder

$$a^2 c + a c^2 - a d^2 - d^3$$

for a dividend, and multiplying by b, to render the division possible by ab, we have

$$\begin{array}{r|l} a^2 bc + abc^2 - abd^2 - bd^3 & ab - ac + d^2 \\ \cline{2-2} - a^2 bc + a^2 c^2 - acd^2 & ac \\ \hline \text{rem. } a^2 c^2 + abc^2 - acd^2 - abd^2 - bd^3, & \end{array}$$

and the term $-ac$ produces still a term $a^2 c^2$, in which a is of the second degree.

To avoid this inconvenience, it must be observed, that the divisor $ab - ac + d^3 = a(b - c) + d^2$, by uniting the terms $ab - ac$ in one; and, for the sake of shortening the operation, making $b - c = m$, we have for the divisor $am + d^2$; but then the whole dividend must be multiplied by the factor m, to make a new dividend, the first term of which may be divided by am, the first term of the divisor; the operation then becomes

$$\begin{array}{r|l} a^2 bm + ac^2 m - d^3 m & am + d^2 \\ \cline{2-2} - a^2 bm - abd^2 & ab + c^2 \\ \hline \text{1st. rem. } - abd^2 + ac^2 m - d^3 m & \\ - ac^2 m - c^2 d^2 & \\ \hline \text{2d rem. } - abd^2 - c^2 d^2 - d^3 m. & \end{array}$$

The terms involving a^2 now disappear from the dividend, and there remain only the terms which have the first power of a. To make these disappear, we first divide the term $a c^2 m$ by $a m$, and it gives for a quotient c^2; multiplying the divisor by this quotient, and subtracting the products from the dividend, we obtain the second remainder. Taking this second remainder for a new dividend, and suppressing the factor d^2, which is not a factor of the divisor, we have

$$-ab - c^2 - dm,$$

which being multiplied anew by m, becomes

$$\begin{array}{rl|l} & -abm - c^2 m - dm^2 & am + d^2 \\ & +abm + bd^2 & -b \\ \hline \text{Rem.} & +bd^2 \quad -c^2 m - dm^2. & \end{array}$$

The remainder $bd^2 - c^2 m - dm^2$ of this last division, not involving a, it follows, that if the proposed quantities have a common divisor, it is independent of the letter a.

Having arrived at this point, we can continue the division no longer with reference to the letter a; but it will be observed, that if there be a common divisor, independent of a, to the quantities $bd^2 - c^2 m - dm^2$ and $am + d^2$, it must divide separately the two parts am and d^2 of the divisor; for if a quantity is arranged with reference to the powers of the letter a, every divisor of this quantity, independent of a, must divide separately the quantities multiplied by the different powers of this letter.

To be convinced of this, we need only observe, that, in this case, each of the quantities proposed must be the product of a quantity depending on a, and of the common divisor, which does not depend upon it. Now ifwe have, for example, the expression

$$Aa^4 + Ba^3 + C^2 a^2 + Da + E,$$

in which the letters A, B, C, D, E, designate any quantities whatever, independent of a, and it be multiplied by a quantity M, also independent of a, the product

$$MAa^4 + MBa^3 + MCa^2 + MDa + ME,$$

arranged with reference to a, will contain still the same powers of a as before; but the coefficient of each of these powers will be a multiple of M.

This being supposed, if we restore the quantity $(b - c)$ in the place of m, we have the quantities

$$bd^2 - c^2(b - c) - d(b - c)^2,$$
$$a(b - c) + d^2;$$

and it is evident, that $b - c$ and d^2 have no common factor; the two quantities then under consideration have not a common divisor.

If it were not evident by mere inspection, that there is no common divisor between $b - c$ and d^2, it would be necessary to seek their greatest common divisor by arranging them with reference to the same letter, and then to see if it would not also divide the quantity

$$b\,d^2 - c^2\,(b - c) - d\,(b - c)^2.$$

50. Instead of putting off to the end of the operation, the investigation of the greatest common divisor independent of the letter with reference to which the quantities are arranged, it is less trouble to seek it at first, because, for the most part, the operation becomes more complicated at each step as we advance, and the process is rendered more difficult.

Let there be, for example, the quantities

$$a^4\,b^2 + a^3\,b^3 + b^4\,c^2 - a^4\,c^2 - a^3\,b\,c^2 - b^2\,c^4,$$
$$a^3\,b + a\,b^2 + b^3 - a^2\,c - a\,b\,c - b^2\,c;$$

having arranged them with reference to the letter a, we have

$$(b^2 - c^2)\,a^4 + (b^3 - b\,c^2)\,a^3 + b^4\,c^2 - b^2\,c^4,$$
$$(b - c)\,a^2 + (b^2 - b\,c)\,a + b^3 - b^2\,c,$$

I observe, in the first place, that if they have a common divisor which is independent of a, it must divide each of the quantities, multiplied by the different powers of a (49), as well as the quantities $b^4\,c^2 - b^2\,c^4$ and $b^3 - b^2\,c$, which do not contain this letter.

The question is reduced then to finding the common divisors of the two quantities $b^2 - c^2$ and $b - c$, and determining whether among these divisors there is to be found one which will divide at the same time

$$b^3 - b\,c^2 \text{ and } b^2 - b\,c,\ b^4\,c^2 - b^2\,c^4 \text{ and } b^3 - b^2\,c.$$

Dividing $b^2 - c^2$ by $b - c$, we find an exact quotient $b + c$; $b - c$ then is a common divisor of the quantities $b^2 - c^2$ and $b - c$, which evidently admit of no other, since the quantity $b - c$ is divisible only by itself and by unity. We must now see whether $b - c$ will divide the other quantities referred to above, or whether it will divide the two quantities proposed; it is found that it will, and it gives

$$(b + c)\,a^4 + (b^2 + b\,c)\,a^3 + b^3\,c^2 + b^2\,c^3,$$
$$a^2 + b\,a + b^2.$$

To bring these last expressions to the greatest degree of simplicity, we should see if the first is not divisible by $b + c$; it appears upon trial that it is, and we have only to find a common divisor to the quantities

$$a^4 + b a^2 + b^2 c^2,$$
$$a^2 + b a + b^2.$$

By proceeding with these as the rule prescribes, we come, after the second division, to a remainder containing the letter a of the first power only; and as this remainder is not the common divisor, we conclude that the letter a does not make a part of the common divisor sought, which therefore can consist only of the factor $b - c$.

If, beside this common divisor, another had been found, involving the quantity a, it would have been necessary to multiply these two divisors together to obtain the common divisor sought.

These remarks will enable the learner, after a little practice in analysis, to find in every case the greatest common divisor. He will determine without difficulty that the quantities.

$$6 a^5 + 15 a^4 b - 4 a^3 c^2 - 10 a^2 b c^2,$$
$$9 a^3 b - 27 a^2 b c - 6 a b c^2 + 18 b c^3,$$

have for their greatest common divisor the quantity $3 a^2 - 2 c^2$.

51. The four *fundamental operations*, addition, subtraction, multiplication and division, we perform in algebra as in arithmetic, observing merely to proceed, in the operations prescribed by the rules of arithmetic, according to the methods given for algebraic quantities. I shall, therefore, merely suggest these methods, giving an example of the application of each. I shall begin as I did in arithmetic, with the multiplication and division of fractions, as they require no preparatory transformations.

1. For multiplication, we have

$$\frac{a}{b} \times c = \frac{ac}{b} \text{ (\textit{Arith.} 53)},$$

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \text{ (\textit{Arith.} 70)}.$$

2. For division,

$$\frac{a}{b} \text{ divided by } c \text{, gives } \frac{a}{bc} \text{ or } \frac{a}{b} \times \frac{1}{c} \text{ (\textit{Arith.} 54, 70)},$$

$$\frac{a}{b} \text{ divided by } \frac{c}{d}\text{, gives } \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc} \text{ (\textit{Arith.} 73)}.$$

3. The fractions $\frac{a}{b}, \frac{c}{d}$, being reduced to the same denominator, become respectively

$$\frac{ad}{bd}, \frac{bc}{bd} \text{ (\textit{Arith.} 79).}$$

The fractions,

$$\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h},$$

by the same reduction, become respectively

$$\frac{adfh}{bdfh}, \frac{cbfh}{bdfh}, \frac{ebdh}{bdfh}, \frac{gbdf}{bdfh}.$$

52. I have given, in art. 79 of arithmetic, a process for obtaining, in certain cases, a denominator more simple, than that which results from the general rule; it may be much simplified by means of algebraic symbols, as we shall see.

If, for example, we have the two fractions $\frac{a}{bc}, \frac{d}{bf}$, it is easy to see that the two denominators would be the same, if f were a factor of the first, and c a factor of the second; we multiply then the two terms of the first fraction by f, and the two terms of the second by c, which gives $\frac{af}{bcf}$ and $\frac{cd}{bcf}$, more simple than $\frac{abf}{bbcf}$ and $\frac{bcd}{bbcf}$, obtained by multiplying by the original denominators.

In general, *to form the common denominator, we collect into one product all the different factors raised to the highest power found in the denominators of the proposed fractions; and it remains only to multiply the numerator of each fraction by the factors of this product, which are wanting in the denominator of the fraction.*

Having, for example, the fractions $\frac{a}{b^2c}, \frac{d}{bf}$, and $\frac{e}{cg}$, I form the product b^2cfg; I multiply the numerator of the first fraction by fg, that of the second by bcg, that of the third by b^2f, and I obtain

$$\frac{afg}{b^2cfg}, \quad \frac{bcdg}{b^2cfg}, \quad \frac{b^2ef}{b^2cfg}.$$

53. The sum of the fractions

$$\frac{a}{d}, \quad \frac{b}{d}, \quad \frac{c}{d},$$

which have the same denominator, or

$$\frac{a}{d} + \frac{b}{d} + \frac{c}{d} = \frac{a+b+c}{d} \text{ (\textit{Arith.} 80).}$$

The difference of the fractions

$$\frac{a}{d} \text{ and } \frac{b}{d},$$

which have the same denominator, or

$$\frac{a}{d} - \frac{b}{d} = \frac{a-b}{d}.$$

The whole of a added to the fraction $\frac{b}{c}$, or the expression

$$a + \frac{b}{c} = \frac{ac}{c} + \frac{b}{c} = \frac{ac+b}{c} \text{ (*Arith.* 81).}$$

Also, the expression

$$a - \frac{b}{c} = \frac{ac}{c} - \frac{b}{c} = \frac{ac-b}{c}.$$

Reciprocally,

the expression $\frac{ac+b}{c} = \frac{ac}{c} + \frac{b}{c} = a + \frac{b}{c}$;

the expression $\frac{ac-b}{c} = \frac{ac}{c} - \frac{b}{c} = a - \frac{b}{c}$.

The terms of the preceding fractions were simple quantities, but if we had fractions, the terms of which were polynomials, we should have to perform, by the rules given for compound quantities, the operations indicated upon simple quantities; it is thus that we have

$$\frac{a^2+b^2}{c+d} \times \frac{a-b}{c-d} = \frac{(a^2+b^2)(a-b)}{(c+d)\ (c-d)} = \frac{a^3+ab^2-a^2b-b^3}{c^2-d^2}.$$

The quotient of the fraction

$$\frac{a^2+b^2}{c+d} \text{ divided by } \frac{a-b}{c-d},$$

is $\frac{a^2+b^2}{c+d} \times \frac{c-d}{a-b} = \frac{(a^2+b^2)(c-d)}{(c+d)\ (a-b)} = \frac{a^2c+b^2c-a^2d-b^2d}{ac+ad-bc-bd}$, and so of other operations.

54. Understanding what precedes, we can resolve an equation of the first degree, however complicated.

If we have, for example, the equation

$$\frac{(a+b)(x-c)}{a-b} + 4b = 2x - \frac{ac}{3a+b},$$

we begin by making the denominators to disappear, indicating only the operations; it becomes then

$$(a+b)(x-c)(3a+b)+4b(a-b)(3a+b) = 2x(a-b)(3a+b)-ac(a-b);$$

performing the multiplications, we have

$$3a^2x + 4abx + b^2x - 3a^2c - 4abc - b^2c + 12a^2b - 8ab^2 - 4b^3 =$$
$$6a^2x - 4abx - 2b^2x - a^2c + abc;$$

transposing to one member all the terms involving x, it becomes

$$-3a^2x+8abx+3b^2x=2a^2c+5abc+b^2c-12a^2b+8ab^2+4b^3,$$

from which we deduce

$$x=\frac{2a^2c+5abc+b^2c-12a^2b+8ab^2+4b^3}{-3a^2+8ab+3b^2}.$$

Of Questions having two Unknown Quantities, and of Negative Quantities.

55. The questions, which we have hitherto considered, involve only one unknown quantity, by means of which, with the known quantities, are expressed all the conditions of the question. It is often more convenient, in some questions, to employ two unknown quantities, but then there must be, either expressed or implied, two conditions, in order to form two equations, without which the two unknown quantities cannot be determined at the same time.

The question in art. 3, especially as it is enunciated in art. 4, presents itself naturally with two unknown quantities, that is, with both the numbers sought. Indeed, if we denote

the least by x,
the greatest by y,
their sum by a,
their difference by b,

we have, by the enunciation of the question,

$$x+y=a,$$
$$y-x=b.$$

Each of these two equations being considered by itself, we can determine one of the unknown quantities. If we take the second, for example, we deduce the value of y, which is

$$y=b+x,$$

a value, which seems at first to teach us nothing with regard to what we are seeking, since it contains the quantity x, which is not given; but if, instead of the unknown quantity y in the first equation, we put this, its equivalent; the equation, containing now only one unknown quantity x, will give the value of x by the process already taught.

We have in fact by this substitution,

$$x+b+x=a,$$

or $$2x+b=a,$$

or lastly, $$x=\frac{a-b}{2};$$

and putting this value of x in the expression for y,

$$y = b + x = b + \frac{a-b}{2} = \frac{a+b}{2};$$

we have then for the two unknown numbers the same expressions as in art. 3.

It is easy to see indeed, that the above solution does not differ essentially from that of art. 3; only I have supposed and resolved the second equation $y - x = b$, which I contented myself with enunciating in common language in the article cited; and from it I deduced, without algebraic calculation, that the greater number was $x + b$.

56. I take another question.

A labourer having worked for a person 12 days, and having with him, during the first 7 days, his wife and son, received 74 francs; he worked afterward with the same person 8 days more, during 5 of which, he had with him his wife and son, and he received at this time 50 francs; how much did he earn per day himself, and how much did his wife and son earn?

Let x be the daily wages of the man,
y that of his wife and son;
12 days' work of the man will amount to $12x$,
7 days' work of his wife and son, $7y$;
we have then by the first statement of the question,

$$12x + 7y = 74;$$

8 days' work of the man will give $8x$,
and 5 days' work of his wife and son $5y$;
we have then by the second statement

$$8x + 5y = 50.$$

Proceeding as in the preceding question, we take the value of y in the first equation, which is

$$y = \frac{74 - 12x}{7},$$

and substitute this value in the second, multiplying it by 5, the coefficient, and it becomes

$$8x + \frac{370 - 60x}{7} = 50,$$

an equation, which contains only the unknown quantity x. By reducing it, we have

$$56x + 370 - 60x = 350,$$
$$370 - 4x = 350;$$

transposing $-4x$ to the second member, and 350 to the first, we obtain

$$\begin{aligned} 370 - 350 &= 4x \\ 20 &= 4x \\ \tfrac{20}{4} &= x \\ 5 &= x. \end{aligned}$$

Knowing x, which we have just found equal to 5, if we place this value in the formula

$$y = \frac{74 - 12x}{7},$$

the second member will be determined, for we have

$$y = \frac{74 - 12 \times 5}{7} = \frac{74 - 60}{7} = \frac{14}{7} = 2;$$

thus $$y = 2.$$

The man then earned 5 francs per day, while his wife and son earned only 2.

57. The reader has perhaps observed, that in resolving the above equation $370 - 4x = 350$, I have transposed $4x$ to the second member. I have proceeded thus to avoid a slight difficulty, that would otherwise have occurred, and which I will now explain.

By leaving $4x$ in the first member, and transposing 370 to the second, we have

$$-4x = 350 - 370;$$

and reducing the second according to the rule in art. 19, there will result from it

$$-4x = -20.$$

But as we have avoided, in the preceding article, the sign $-$, which affects the quantity $4x$, by transposing this quantity to the other member; and as in like manner the quantity $350 - 370$ becomes by transposition $370 - 350$; and since a quantity, by being thus transferred from one member to the other, changes the sign (10), it is evident that we may come to the same result by simply changing the sign of each of the quantities $-4x$, $+350 - 370$, which gives

$$4x = -350 + 370,$$

or $$4x = 370 - 350,$$

which is the same as

$$370 - 350 = 4x.$$

We might also change the signs after reduction, and the equation

$$-4x = -20$$

becomes, as above,

$$4x = 20.$$

It follows from this, that *we may transpose indifferently, to one member or to the other, all the terms involving the unknown quantity, observing merely to change the signs of the two members in the result, when the unknown quantity has the sign* —.

58. Having undertaken, by means of letters, a general solution of the problem of art. 56, I will now examine a particular case. I suppose the first sum received by the labourer to be 46 francs, and the second 30, the other circumstances remaining as before; the equations of the question will then be

$$12x + 7y = 46,$$
$$8x + 5y = 30.$$

The first gives

$$y = \frac{46 - 12x}{7};$$

multiplying this value by 5, in order to substitute it in the place of $5y$, in the second, we have

$$8x + \frac{230 - 60x}{7} = 30;$$

the denominator being made to disappear, it becomes

$$56x + 230 - 60x = 210,$$

or

$$56x - 60x = 210 - 230,$$

or

$$-4x = -20,$$

and the signs being changed agreeably to what has just been remarked,

$$4x = 20,$$
$$x = \tfrac{20}{4} = 5.$$

If we substitute this value instead of x in the expression for y, it will become

$$y = \frac{46 - 60}{7},$$

or

$$y = \frac{-14}{7}.$$

Now how are we to interpret the sign —, which affects the insulated quantity 14? We understand its import, when there are two quantities separated from each other by the sign —, and

when the quantity to be subtracted is less than that from which it is to be taken; but how can we subtract a quantity when it is not connected with another in the member where it is found? To clear up this difficulty, it is best to go back to the equations, which express the conditions of the question; for the nearer we approach to the enunciation, the closer shall we bring together the circumstances which have given rise to the present uncertainty.

I resume the equation

$$12x + 7y = 46;$$

I put in the place of x its value 5, and it becomes

$$60 + 7y = 46.$$

This equation, by mere inspection, presents an absurdity. It is impossible to make the number 46 by adding any thing to the number 60, which exceeds it already.

I take also the second equation,

$$8x + 5y = 30,$$

and putting 5 in the place of x, I find

$$40 + 5y = 30;$$

the same absurdity as before, since the number 30 is to be formed by adding something to the number 40.

Now the quantities $12x$ or 60 in the first equation, $8x$ or 40 in the second, represent what the labourer earned by his own work; the quantities $7y$ and $5y$ stand for the earnings of his wife and son, while the numbers 46 and 30 express the sum given as the common wages of the three; we must see then at once in what consists the absurdity.

According to the question, the labourer earned more by himself, than he did by the assistance of his wife and son; it is impossible then to consider what is allowed to the woman and son, as augmenting the pay of the labourer.

But if, instead of counting the allowance made to the two latter persons as positive, we regard it as a charge placed to the account of the labourer, then it would be necessary to deduct it from his wages; and the equations would no longer involve a contradiction, as they would become

$$60 - 7y = 46,$$
$$40 - 5y = 30;$$

we deduce from the one as well as from the other

$$y = 2;$$

and we conclude from it, that if the labourer earned 5 francs per day, his wife and son were the occasion of an expense of 2 francs, which may otherwise be proved thus.

For 12 days' labour he received

$$5 \times 12 \text{ or } 60 \text{ francs};$$

the expense of his wife and son for 7 days is

$$2 \times 7 \text{ or } 14 \text{ francs};$$

there remain then 46 francs.

For 8 days' labour he receives

$$5 \times 8 \text{ or } 40 \text{ francs};$$

the expense of his wife and son for 5 days is

$$2 \times 5 \text{ or } 10 \text{ francs},$$

there remain 30 francs.

It is very clear then, that in order to render the proposed problem with the first conditions possible, instead of the enunciation in article 56, we must substitute this;

A labourer worked for a person 12 *days, having had with him the first* 7 *days, his wife and son at a certain expense, and he received* 46 *francs; he worked afterwards* 8 *days, during* 5 *of which he had with him his wife and son at an expense as before, and he received* 30 *francs. It is required to find how much he earned per day, and what was the sum charged him per day on account of his wife and son.*

Calling x the daily wages of the labourer, and y the daily expense of wife and son, the equations of the problem will evidently be

$$12x - 7y = 46,$$
$$8x - 5y = 30;$$

and being resolved after the manner of those in art. 56, they will give

$$x = 5 \text{ francs}, \quad y = 2 \text{ francs}.$$

59. In every case, where we find, for the value of the unknown quantity, a number affected with the sign —, we can rectify the enunciation in a manner analogous to the preceding, by examining, with care, what that quantity is, among those, which are additive in the first equation, which ought to be subtractive in the second; but algebra supersedes the use of every inquiry of this kind, when we have learnt to make a proper use of expressions affected with the sign —; for these expressions, being deduced from the equations of the problem, must *satisfy* those

equations; that is to say, by subjecting them to the operations indicated in the equation, we ought to find for the first member a value equal to that of the second. Thus the expression $\frac{-14}{7}$ drawn from the equations

$$12x + 7y = 46,$$
$$8x + 5y = 30,$$

must, consistently with the value of $x = 5$, as deduced from these same equations, verify them both.

The substitution of the value of x gives, in the first place,

$$60 + 7y = 46,$$
$$40 + 5y = 30.$$

It remains to make the substitution of $\frac{-14}{7}$ in the place of y; and for this purpose we must multiply by 7 and by 5, having regard to the sign —, with which the numerator of the fraction is affected.

If we apply the rule relative to the signs given in art. 42 for division, we have

$$\frac{-14}{7} = -2;$$

besides, by the rule for the signs in multiplication, we find

$$7 \times -2 = -14,$$
$$5 \times -2 = -10.$$

Hence the equations

$$60 + 7y = 46, \quad \text{and} \quad 40 + 5y = 30,$$

become respectively

$$60 - 14 = 46, \quad \text{and} \quad 40 - 10 = 30,$$

and are verified, not by adding the two parts of the first member, but in reality by subtracting the second from the first, as was done above, after considering the proper import of the equations.

60. The problem in art. 58 does not admit of a solution in the sense in which it is first enunciated; that is to say, by addition, or regarding as an accession the sum considered with reference to the wife and son of the labourer; neither does the second enunciation consist with the data of the problem in art. 56.

If we were to consider in this case y, as expressing a deduction, the equations thus obtained

$$12x - 7y = 74,$$
$$8x - 5y = 50,$$

would give

$$x = 5, \quad \text{and} \quad y = \frac{-14}{7};$$

and the substitution of the value of x would immediately change the equations to

$$60 - 7y = 74,$$
$$40 - 5y = 50.$$

The absurdity of these results is precisely contrary to that of the results in art. 58, since it relates to remainders greater than the numbers 60 and 40, from which the quantities $7y$ and $5y$ are to be subtracted.

The sign minus, which belongs to the expression of y, implies an absurdity; but this is not all, it does it away also; for, according to the rule for the signs,

$$\frac{-14}{7} = -2,$$

and

$$-7 \times -2 = +14,$$
$$-5 \times -2 = +10.$$

Thus the equations

$$60 - 7y = 74, \qquad 40 - 5y = 50,$$

become

$$60 + 14 = 74, \qquad 40 + 10 = 50,$$

and are verified by addition; consequently, the quantities $-7y$ and $-5y$, transformed into $+14$, $+10$, instead of expressing expenses incurred by the labourer, are regarded as a real gain. We are brought back then, in this case, also to the true enunciation of the question.

61. We perceive by the preceding examples, that *there may be, in the enunciations of a problem of the first degree, certain contradictions, which algebra not only makes known, but points out also how they may be reconciled, by rendering subtractive certain quantities which had been regarded as additive, or additive certain quantities which had been regarded as subtractive, or by giving to the unknown quantities values affected with the sign* —.

See then what is to be understood, when we speak of values affected by the sign —, and of what are called *negative solutions*, resolving, in a sense opposite to the enunciation, the question in which they occur.

It follows from this, that we may regard, as but one single question, those, the enunciations of which are connected together in such a manner, that the solutions, which satisfy one of the enunciations, will, by a mere change of sign, satisfy the other also.

62. Since negative quantities resolve in a certain sense the problems, which give rise to them, it is proper to inquire a little more particularly into the use of these quantities, and to settle once for all the manner of performing operations in which they are concerned.

We have already made use of the rule for the signs, which had been previously determined for each of the fundamental operations; but the rules have not been demonstrated with reference to insulated quantities. In the case of subtraction, for example, we supposed that there was to be taken from a, the expression $b - c$, in which the negative quantity c was preceded by a positive quantity b. Strictly speaking, the reasoning does not depend upon the value of b; it would still apply when $b = 0$, which reduces the expression $b - c$ to $-c$. But the theory of negative quantities being at the same time one of the most important and most difficult in algebra, it should be established upon a sure basis. To effect this, it is necessary to go back to the origin of negative quantities.

The greatest subtraction, that can be made from a quantity, is to take away the quantity itself, and in this case we have zero for a remainder; thus $a - a = 0$. But when the quantity to be subtracted exceeds that from which it is to be taken, we cannot subtract it entirely; we can only make a reduction of the quantity to be subtracted, equal to the quantity from which it was to be taken. When, for example, it is required to subtract 5 from 3, or when we have the quantity $3 - 5$; to take, in the first place, 3 from 5, we decompose 5 into two parts 3 and 2, the successive subtraction of which will amount to that of 5, and thus, instead of $3 - 5$, we have the equivalent expression $3 - 3 - 2$, which is reduced to -2. The sign $-$, which precedes 2, shows what is necessary to complete the subtraction; so that, if we had added 2 to the first of the quantities, we should have had $3 + 2 - 5$, or zero. We express then, with the help of algebraic signs, the idea that is to be attached to a negative quantity $-a$, by forming the equation $a - a = 0$, or by regarding the symbols $a - a$, $b - b$, &c., as equivalent to zero.

This being supposed, it will be understood, that if we add to any quantity whatever the symbol $b - b$, which in reality is only zero, we do not change the value of this quantity, and that, consequently, the expression $a + b - b$, is nothing else but a different manner of writing the quantity a, which is also evident from the consideration, that $+ b$ and $- b$ destroy each other.

But having by this change of form introduced $+ b$ and $- b$ into the same expression with a, we see, that in order to subtract any one of these quantities, it is sufficient to efface it. If it were $+ b$ that we would subtract, we efface it, and there remains $a - b$, which accords with the rule laid down in art. 2; if on the other hand it were $- b$, we efface this quantity, and there would remain $a + b$, as might be inferred from art. 20.

With respect to multiplication, it will be observed, that the product of $a - a$ by $+ b$ must be $a b - a b$, because the multiplicand being equal to zero, the product must be zero; and the first term being $a b$, the second must necessarily be $- a b$ to destroy the first.

We infer from this, that $- a$, multiplied by $+ b$, must give $- a b$.

By multiplying a by $b - b$, we have still $a b - a b$, because the multiplier being equal to zero, the product will also be equal to zero; it is therefore necessary that the second term should be $- a b$, to destroy the first $+ a b$.

Whence $+ a$, multiplied by $- b$, must give give $- a b$.

Lastly, if we multiply $- a$ by $b - b$, the first term of the product being, according to what has just been proved, $- a b$, it is necessary that the second term should be $+ a b$, as the product must be nothing when the multiplier is nothing.

Whence $- a$, multiplied by $- b$, gives $+ a b$.

By collecting these results together we may deduce from them the same rules as those in art. 31 (A).

As the sign of the quotient, combined with that of the divisor, according to the rules proper for multiplication, must produce the sign of the dividend, we infer from what has just been said, that the rule for the signs given in art. 42, corresponds with that, which it is necessary to observe in fact, and that consequently, *simple quantities, when they are insulated, are combined with respect to their signs, in the same manner, as when they make a part of polynomials.*

63. According to these remarks, we may always, when we meet with negative values, go back to the true enunciation of the question resolved, by seeking in what manner these values will satisfy the equations of the proposed problem; this will be confirmed by the following example, which relates to numbers of a different kind from those of the question in art. 56.

64. *Two couriers set out to meet each other at the same time from two cities, the distance of which is given; we know how many miles* (a) *each travels per hour, and we inquire at what point of the route between the two cities they will meet.*

To render the circumstances of the question more evident, I have subjoined a figure, in which the points A and B represent the places of departure of the couriers.

A R B

I denote the things given, and those required, in the usual way, by small letters.

a, the distance in miles of the points of departure A and B,

b, the number of miles per hour, which the courier from A travels,

c, the number of miles per hour, which the courier from B travels.

The letter R being placed at the point of meeting of the two couriers, I shall call x the distance AR passed over by the first, y the distance BR passed over by the second, and as

$$AR + BR = AB,$$

I have the equation,

$$x + y = a.$$

Considering that the distances x and y are passed over in the same time, we remark that the first courier, who travels a number b of miles in an hour, will employ, in passing over the distance x, a time denoted by $\frac{x}{b}$.

Also the second courier, who travels c miles in an hour, will employ, in passing over the distance y, a time denoted by $\frac{y}{c}$; we have then

$$\frac{x}{b} = \frac{y}{c}.$$

(a) In the original the distance is given in *kilometres*. It is here expressed by miles to avoid perplexing the learner.

The equations of the question therefore will be

$$x + y = a,$$

$$\frac{x}{b} = \frac{y}{c}.$$

Making the denominator b of the second to disappear, we have

$$x = \frac{by}{c};$$

putting this value in the place of x in the first equation, it becomes

$$\frac{by}{c} + y = a;$$

and we deduce from it

$$by + cy = ac, \qquad \text{whence} \qquad y = \frac{ac}{b+c}.$$

Substituting this value of y in the expression for the value of x, we obtain

$$x = \frac{b}{c} \times \frac{ac}{b+c}, \qquad \text{or} \qquad x = \frac{abc}{c(b+c)} \text{ (51)},$$

or lastly

$$x = \frac{ab}{b+c} \text{ (38)}.$$

As the sign — does not enter into the values of x and y, it is evident that whatever numbers are put for the letters $a\,b\,c$, we shall always find x and y with the sign $+$, and therefore the question proposed will be resolved in the precise sense of the enunciation. Indeed it is readily perceived, that in every case where two persons set off from different points and travel toward each other, they must necessarily meet.

65. I will now suppose, that the two couriers proceed in the same direction, and that the one who sets out from *A* is pursuing the one who sets out from *B*, and who is travelling toward the same point *C*, placed beyond *B*, with respect to *A*.

A B R C

It is evident that in this case, the courier who starts from the point *A*, cannot come up with the courier who sets off from the point *B*, except he travels faster than this last, and the point of coming together, *R*, cannot be between *A* and *B*, but must be beyond *B*, with respect to *A*.

Having the same things given as before, and observing that when

$$AR - BR = AB,$$

we have

$$x - y = a.$$

The second equation,

$$\frac{x}{b} = \frac{y}{c},$$

expressing only the equality of the times employed by the couriers in passing over the distances *AR* and *BR*, undergoes no change.

The above equations, being resolved like the former ones, give

$$x = \frac{by}{c},$$

$$\frac{by}{c} - y = a, \qquad by - cy = ac,$$

$$y = \frac{ac}{b-c},$$

$$x = \frac{b}{c} \times \frac{ac}{b-c} = \frac{abc}{c(b-c)},$$

and lastly $$x = \frac{ab}{b-c}.$$

Here the values of x and y will not be positive, except when b is taken greater than c, that is to say, except the courier starting from the point *A* be supposed to travel faster than the other.

If, for example, we make

$$b = 20, \qquad c = 10,$$

we have

$$x = \frac{20\,a}{20 - 10} = \frac{20\,a}{10} = 2\,a,$$

$$y = \frac{10\,a}{20 - 10} = \frac{10\,a}{10} = a;$$

from which it follows, that the point of their coming together is distant from the point *A* twice *AB*.

If we now suppose b smaller than c, and take, for example,

$$b = 10, \qquad c = 20,$$

we find

$$x = \frac{10\,a}{10 - 20} = \frac{10\,a}{-10} = -a,$$

$$y = \frac{20\,a}{10 - 20} = \frac{20\,a}{-10} = -2\,a.$$

These values being affected with the sign —, make it evident, that the question cannot be resolved in the sense in which it is enunciated; and indeed it is absurd to suppose that the courier

setting out from the point A, and proceeding only 10 miles in an hour, should ever be able to overtake the courier setting out from the point B, and travelling 20 miles per hour, and who is in advance of the first.

66. Nevertheless, these same values resolve the question in a certain sense; for, by substituting them in the equations

$$x - y = a,$$

$$\frac{x}{b} = \frac{y}{c},$$

we have by the rule for the signs

$$-a + 2a = a,$$

$$-\frac{a}{10} = -\frac{2a}{20},$$

equations which are satisfied; since, by making the reductions, the first member becomes equal to the second; and if we attend to the signs of the terms, which compose the first, we shall see how it is necessary to modify the enunciation of the question, in order to do away the absurdity.

Indeed, it is the distance a corresponding to x, and passed over by the first courier, which is in reality subtracted from the distance $2a$, corresponding to y, and passed over by the second courier; it is then just as if we had changed y into x, and x into y, and had supposed that the courier starting from the point B, had run after the other.

This change in the enunciation, produces also a change in the direction of the routes of the couriers; they are no longer travelling toward the point C, but in an opposite manner toward the point C', as represented in the figure below;

C′ R′ A B R C

and their coming together takes place in R'. The result from this is

$$BR' - AR' = AB,$$

which gives

$$y - x = a;$$

we have besides constantly

$$\frac{x}{b} = \frac{y}{c},$$

and we find

$$x = \frac{ab}{c-b} = \frac{10a}{20-10} = a,$$

$$y = \frac{ac}{c-b} = \frac{20a}{20-10} = 2a,$$

positive values, which resolve the question in the precise sense in which it is enunciated.

67. The question we have been considering presents a case, in which it is in every sense absurd. This occurs when we suppose the two couriers to travel equally fast. It is evident, that in whatever direction we suppose them to move, they can never come together, since they preserve constantly the interval of their points of departure. This absurdity, which no modification in the enunciation can remove, is very conspicuous in the equations.

We have now $b = c$, since the couriers, travelling equally fast, pass over the same space in an hour; the equation

$$\frac{x}{b} = \frac{y}{c}$$

becomes
$$\frac{x}{b} = \frac{y}{b},$$

and gives $x = y$.
Thus the equation $x - y = a$
reduces itself to $x - x = a$ or $0 = a$,
a result sufficiently absurd, since it supposes a quantity a, the magnitude of which is given, to be nothing.

68. This absurdity shews itself in a manner very singular in the values of the unknown quantities

$$x = \frac{ab}{b-c}, \qquad y = \frac{ac}{b-c};$$

their denominator becoming 0 when $b = c$, we have

$$x = \frac{ab}{0}, \qquad y = \frac{ac}{0}.$$

We do not easily perceive what may be the quotient of a division when the divisor is zero; we see merely, that if we consider b as nearly equal to c, the values of x and y become very great. To be convinced of this, we need only take

$$b = 6 \text{ miles}, \qquad c = 5{,}8 \text{ miles},$$

we then have
$$x = \frac{6a}{0{,}2} = 30a,$$

$$y = \frac{5{,}8a}{0{,}2} = 29a.$$

If further we take $b = 6$, $c = 5{,}9$,

we have

$$x = \frac{6\,a}{0{,}1} = 60\,a,$$

$$y = \frac{5{,}9\,a}{0{,}1} = 59\,a.$$

If moreover we make

$$b = 6, \qquad c = 5{,}99,$$

it becomes

$$x = \frac{6\,a}{0{,}01} = 600\,a,$$

$$y = \frac{5{,}99\,a}{0{,}01} = 599\,a,$$

and it is manifest, that as the divisor diminishes in proportion to the smallness of the assumed difference of the numbers b and c, we obtain values more and more increased in magnitude.

But as a quantity, however minute, can never be taken for zero, it follows, that however small we make the difference of the numbers represented by the letters b and c, and however great may be the consequent values of x and y, we never attain to those which answer to the case where $b = c$.

Since these last cannot be represented by any number, however great we suppose it, they are said to be *infinite;* and every expressión of the form $\frac{m}{0}$, the denominator of which is zero, is regarded as the symbol of *infinity.*

This example shows that mathematical *infinity* is a negative idea, since we at length get it only by the impossibility of assigning a quantity that can resolve the question.

We may ask here, how the values

$$x = \frac{a\,b}{0}, \qquad y = \frac{a\,c}{0},$$

satisfy the equations proposed; for it is an essential characteristic of algebra, that the symbols of the values of unknown quantities, whatever they may be, being subjected to the operations indicated upon these quantities, shall satisfy the equations of the problem.

By substituting them in the equations

$$x - y = a,$$

$$\frac{x}{b} = \frac{y}{b},$$

which answer to the case where $b = c$, we have by the first,

$$\frac{ab}{0} - \frac{ab}{0} = a,$$

or $\quad \frac{ab - ab}{0} = a, \quad$ or $\quad ab - ab = a \times 0,$

or lastly, $\quad 0 = 0, \quad$ since $\quad a \times 0 = 0.$

The second equation gives, under the same condition,

$$\frac{ab}{0 \times b} = \frac{ab}{0 \times b};$$

the two members of each equation becoming equal, the equations are satisfied.

It remains still to explain how the notion indicated by the expression $\frac{ab}{0}$, removes the absurdity of the result found in art. 67. For this purpose, let the two members of the equation

$$x - y = a,$$

be divided by x, which gives

$$1 - \frac{y}{x} = \frac{a}{x};$$

and as the equation

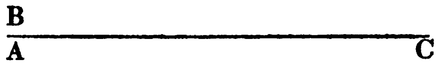
$$\frac{x}{b} = \frac{y}{b}$$

gives $x = y$, the first becomes

$$1 - 1 = \frac{a}{x}, \qquad \text{or} \qquad 0 = \frac{a}{x}.$$

The error here consists in the quantity $\frac{a}{x}$, by which the second member exceeds the first; but this error becomes smaller and smaller, in proportion to the assumed magnitude of x. It is then with reason, that algebra gives for x an expression, which cannot be represented by any number, however great, but which, as it proceeds in the order of numbers becoming greater and greater, points out in what manner we may reduce more and more the error of the supposition.

69. If the couriers travelling equally fast, and in the same direction, had set out from the same point, their coming together could not be said to take place at any particular point, since they would be together through the whole extent of their route. It may be worth while to see how this circumstance is represented by the values, which the unknown quantities x and y assume in this case.



The points A and B being coincident, we have on this supposition $a = 0$, and constantly $b = c$; it follows then, that

$$x = \frac{0.b}{0} = \frac{0}{0}, \quad y = \frac{0.c}{0} = \frac{0}{0}.$$

In order to interpret these values, that indicate a division, in which the dividend and divisor are each nothing, I go back to the equations of the question. The first becoming

$$x - y = 0, \quad \text{gives} \quad x = y;$$

and substituting this value in the second equation, which is

$$\frac{x}{b} = \frac{y}{b}, \quad \text{it becomes} \quad \frac{y}{b} = \frac{y}{b}.$$

The last equation having its two members *identical*, that is to say, composed of the same terms with the same sign, is verified, whatever value is assigned to y, and this unknown quantity can never be determined. Besides, it is evident that the equation

$$\frac{x}{b} = \frac{y}{b} \quad \text{becomes } x = y,$$

and consequently can express nothing more than the first.* The only result, both from the one and from the other, is, that the two couriers are always together, since the distances x and y from the point A are equal; their value in other respects remains indeterminate. The expression $\frac{0}{0}$ then, is here a symbol of an indeterminate quantity. I say here, for there are cases where it is not; but the expression has not then the same origin as the preceding.

70. To give an example, let there be

$$\frac{a(a^2 - b^2)}{b(a - b)}.$$

This quantity becomes $\frac{0}{0}$ in its present form, when $a = b$; but if we reduce it first to its most simple expression, by suppressing the factor $a - b$, common to the numerator and denominator, we find

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* For the sake of conciseness, analysts apply to the same equations the epithet, *identical*.

$\frac{y}{b} = \frac{y}{b}$ is an identical equation, $5 - 3x = 5 - 3x$ is another, and when two equations express only the same thing, we say that these equations also are identical.

$$\frac{a(a+b)}{b},$$

which gives $2a$, when $a = b$.

It is not the same with the values of x and y, found in the preceding article, for they are not susceptible of being reduced to a more simple expression.

It follows, from what I have just said, that when we meet with an expression which becomes $\frac{0}{0}$, it is proper, before pronouncing upon its value, to see if the numerator and denominator have not a common factor, which becoming nothing, renders the two terms at the same time equal to zero, and which being suppressed, the true value of the proposed expression is obtained. There are, notwithstanding, some cases which elude this method, but the limits of this work will only allow me to note the *analytical fact.* It belongs properly to the differential calculus, to give the general processes for finding the true value of quantities, which become $\frac{0}{0}$.

71. It is very evident, from what has been said, that *algebraic solutions either answer perfectly to the conditions of a problem, when it is possible, or they indicate a modification to be made in the enunciation, when the things given imply contradictions that cannot be reconciled; or lastly, they make known an absolute impossibility, when there is no method of resolving with the same things given, a question analogous in a particular sense to the one proposed.*

72. It may be remarked, that in the different solutions of the preceding question, the changing of the signs of the unknown quantities x and y, corresponds to a change in the direction of the journeys represented by the unknown quantities. When the unknown quantity y was counted from B towards A, it had in the equation

$$x + y = a,$$

the sign $+$, and it takes the sign $-$ for the second case, when the motion is in the opposite direction from B towards C, art. 65, since we had for the first equation

$$x - y = a.$$

By changing the sign in the second equation,

$$\frac{x}{b} = \frac{y}{c},$$

we have

$$\frac{x}{b} = \frac{-y}{c},$$

a result which differs from that given in the article cited; but it should be observed, that the journey y, being made up of multiples of the space c passed over in an hour by the courier from B, and this space having the same direction as the space y, ought to be supposed to have the same sign, and consequently to take the sign —, when — is applied to y; we have accordingly,

$$\frac{x}{b}=\frac{-y}{-c}, \text{ or } \frac{x}{b}=\frac{y}{c}.$$

A simple change of sign then is sufficient to comprehend the second case of the question in the first, and it is thus that algebra gives at the same time the solution of several analogous questions.

We have a striking example of this in the problem of art. 15. It is here supposed that the father owed the son a sum d; if we would resolve the question on the contrary hypothesis, that is, by supposing that the son owed the father the sum d, it would be sufficient to change the sign of d in the value of x, and we have

$$x=\frac{bc-d}{a+b}.$$

If we suppose neither to owe the other any thing, we must make $d=0$, and then the equation would be

$$x=\frac{bc}{a+b}.$$

Nothing can be easier than to verify the two solutions, by putting anew the problem into an equation for each of the cases, which we have enunciated.

73. It was only to preserve an analogy between the problems 56 and 64, that I have employed two unknown quantities in the second. Each may be resolved with only one unknown quantity; for when we say that the labourer received 74 francs for 12 days' work performed by himself and 7 days' work by his wife and son, it follows, that if we call y the daily wages of the woman and son, and take $7y$ from 74 francs, there will remain $74-7y$ for the 12 days' labour of the man; from which we infer that he earned $\frac{74-7y}{12}$ per day.

By a similar calculation for the 8 days' service, we find that he earned $\frac{50-5y}{8}$ per day.

Putting the two quantities equal to each other, we form the equation

$$\frac{74-7y}{12}=\frac{50-5y}{8}.$$

Also in the question of art. 64,

A R B

if x represent the course AR of the courier from A, $BR = a - x$ would be that of the courier who set off from B towards A. These two distances being passed over in the same time by the couriers, whose rate of travelling per hour in miles is denoted by numbers b and c respectively, we have

$$\frac{x}{b}=\frac{a-x}{c},$$

whence

$$cx = ab - bx,$$

$$x=\frac{ab}{b+c}.$$

The difference between the solutions, which I have now given, and those of articles 56 and 64, consists merely in this, that we have formed and resolved the first equation by the assistance of ordinary language, without employing algebraic characters, and it is manifest, that the further we carry this, the less will remain to be effected by the other.

74. We sometimes add to the problem of art. 64 a circumstance, which does not render it more difficult.

A R C B

We suppose that the courier, who starts from B, *sets off a number* d *of hours before the other, who goes from* A.

It is evident, that this amounts only to a change of the point of departure of the first, for if he travelled a number c of miles per hour, he would pass over the space $BC = cd$ in d hours, and would be at the point C, when the other courier set off from A; so that the interval of the points of departure would be

$$AC = AB - BC = a - cd.$$

By writing then $a - cd$ in the place of a in the equation of the preceding article, we have

$$\frac{x}{b}=\frac{a-cd-x}{c},$$

$$x=\frac{ab-bcd}{b+c}.$$

If the couriers proceeded in the same direction, the interval of

A B C R

the points of departure would be

$$AC = AB + BC = a + cd;$$

and the distance passed over by the courier from the point A would be AR, while that passed over by the other courier would be $$CR = AR - AC;$$
we have then

$$\frac{x}{b} = \frac{x - a - cd}{c},$$

whence

$$x = \frac{ab + bcd}{b - c}.$$

75. Enunciated in this manner, the problem presents a case, in which the interpretation of the negative value found for x is attended with some difficulty; it is when the couriers being supposed to proceed in opposite directions, we give to the number d a value such, that the space BC represented by cd, becomes greater than a, which represents AB.

C R A B

Now the courier from the point B arrives at C on the other side of A at the moment when the courier from A sets off towards B; there is then an absurdity in supposing that the two couriers can thus come together.

If we should take, for example,

$$a = 400^{\text{mls.}},\quad b = 12^{\text{mls.}},\quad c = 8^{\text{mls.}},\quad d = 60^{\text{h}},$$

there would result from it $cd = 480^{\text{mls.}}$, thus the point C would be $80^{\text{mls.}}$ on the other side of A, with respect to the point B; but we find,

$$x = \frac{400 \,.\, 12 - 60 \,.\, 8 \,.\, 12}{8 + 12} = \frac{400 \,.\, 3 - 60 \,.\, 2 \,.\, 12}{2 + 3}$$
$$= \frac{1200 - 1440}{5} = -\frac{240}{5} = -48.$$

Thus the coming together of the couriers takes place in a point R, $48^{\text{mls.}}$ on the other side of the point A, but between A and C; although it seems that the courier from B, being supposed to continue his journey beyond the point C, can be overtaken by the other courier only after he has passed this point.

To understand the question resolved in this sense, we may substitute in the place of x the negative member $-m$, and the

equation becomes

$$-\frac{m}{b}=\frac{a-cd+m}{c},$$

or by changing the signs in the two members,

$$\frac{m}{b}=\frac{cd-a-m}{c}.$$

We see that the distance passed over by the courier from the

. . . .——————————

C R A B

point B, is $cd-a-m$, or what remains of BC after AB and AR are subtracted, that is CR, and that $AC=cd-a$. This is just what would take place if the second courier had started immediately from the point C, where he is, at the departure of the first; but as they travel in opposite directions, they must necessarily meet between A and C. Thus, this case is similar to the first of those of art. 74, where it is sufficient to change $a-cd$ into $cd-a$, in order to obtain the value, which m has according to the above equation.*

76. The problem of art. 56, taken in its most enlarged sense, may be enunciated as follows;

A labourer having passed a number a *of days in a family, and having with him his wife and son during a number* b *of days, received a sum* c*; he lived afterward in the same family a number* d *of days; he had with him this time his wife and son, during a number* e *of days, and he received a sum* f*; we inquire what he earned per day, and what was allowed per day to his wife and son.*

Let x represent constantly the daily wages of the labourer, and y that of his wife and son; for the number a of days, he has ax, and for the number b of days, his wife and son have by, so that,

$$ax+by=c;$$

for the number d of days, he has dx, and for the number e of days, his wife and son have ey, thus,

$$dx+ey=f.$$

These are the general equations of the question.

We deduce from the first

$$x=\frac{c-by}{a};$$

multiplying this value by d, in order to substitute it in the place

* See note at the end of the Elements of Algebra.

of x in the second equation, we have

$$dx = \frac{cd - bdy}{a},$$

and consequently,

$$\frac{cd - bdy}{a} + ey = f.$$

By making the denominator to disappear, we obtain

$$cd - bdy + aey = af,$$

whence

$$aey - bdy = af - cd,$$

$$y = \frac{af - cd}{ae - bd}.$$

Having the value of y, if we substitute it instead of y in the expression for x, this last will be known,

$$x = \frac{c - b\,\dfrac{af - cd}{ae - bd}}{a}.$$

To simplify this expression, we should, in the first place, perform the multiplication indicated upon the quantities

$$b, \quad \text{and} \quad \frac{af - cd}{ae - bd}, \qquad (51)$$

which gives

$$x = \frac{c - \dfrac{abf - bcd}{ae - bd}}{a};$$

and then reduce c to a fraction having the same denominator as the fraction which accompanies it, and perform the subtraction of this fraction (53); and it becomes

$$x = \frac{\dfrac{ace - bcd - abf + bcd}{ae - bd}}{},$$

or by being reduced

$$x = \frac{\dfrac{ace - abf}{ae - bd}}{a}.*$$

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* There might be some doubt as to the meaning of this expression; but it is obviated by attending to the bar denoting division, which is placed in the middle of the line. Thus, in the expression $x = \frac{A}{B}$, A represents the dividend, whether integral or fractional, and B the divisor, which may also be a whole number or a fraction. So also

Dividing by a (51) we have

$$x = \frac{ace - abf}{a^2 e - abd}.$$

Suppressing the factor a, common to the numerator and denominator (38), we find

$$x = \frac{ce - bf}{ae - bd}.$$

The values

$$x = \frac{ce - bf}{ae - bd}, \quad y = \frac{af - cd}{ae - bd},$$

are applied in the same manner as those, which we before found for literal equations, with only one unknown quantity; we substitute in the place of the letters, the particular numbers in the example selected.

We shall obtain the results in art. 56, by making

$$a = 12, \quad b = 7, \quad c = 74,$$
$$d = 8, \quad e = 5, \quad f = 50,$$

and those of art. 58, by making

$$a = 12, \quad b = 7, \quad c = 46,$$
$$d = 8, \quad e = 5, \quad f = 30.$$

77. The values of x and y are adapted not only to the proposed question; they extend also to all those, which lead to two equations of the first degree with two unknown quantities, since it is evident, that these equations are necessarily comprehended in the formulas,

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the expression $x = \dfrac{\frac{A}{C}}{B}$ signifies, that x is equal to the quotient of the fraction $\dfrac{A}{C}$ divided by B, and the expression $x = \dfrac{A}{\frac{B}{C}}$ indicates for x the quotient arising from A divided by the fraction $\dfrac{B}{C}$; and lastly, we denote by the expression $x = \dfrac{\frac{A}{C}}{\frac{B}{D}}$, the quotient resulting from the division of the fraction $\dfrac{A}{C}$ by the fraction $\dfrac{B}{D}$.

It will be perceived by these remarks, that it is necessary to place the bars according to the result, which we propose to express.

$$ax + by = c,$$
$$dx + ey = f,$$

provided the letters a, b, d, e, denote the whole of the given quantities, by which the unknown quantities x and y are respectively multiplied, and the letters c and f, the whole of the known terms, transposed to the second member.

Of the resolution of any given number of Equations of the First Degree, containing an equal number of unknown Quantities.

78. When a question has as many distinct conditions, as it contains unknown quantities, each of these conditions furnishes an equation, in which it often happens, that the unknown quantities are involved with others, as we have seen already in the problems with two unknown quantities; but if these unknown quantities are only of the first degree, according to the method adopted in the preceding articles, *we take in one of the equations the value of one of the unknown quantities, as if all the rest were known, and substitute this value in all the other equations, which will then contain only the other unknown quantities.*

This operation, by which we exterminate one of the unknown quantities, is called *elimination*. In this way, if we have three equations with three unknown quantities, we deduce from them two equations with only two unknown quantities, which are to be treated as above; and having obtained the values of the two last unknown quantities, we substitute them in the expression for the value of the first unknown quantity.

If we have four equations with four unknown quantities, we deduce from them, in the first place, three equations with three unknown quantities, which are to be treated in the manner just described; having found the value of the three unknown quantities, we substitute them in the expression for the value of the first, and so on.

See an example of a question, which contains three unknown quantities and three equations.

79. *A person buys separately three loads of grain; the first, which contained* 30 *measures of rye,* 20 *of barley, and* 10 *of wheat, cost* 230 *francs;*

The second, which contained 15 *measures of rye,* 6 *of barley, and* 12 *of wheat, cost* 138 *francs;*

The third, which contained 10 *measures of rye,* 5 *of barley, and* 4 *of wheat, cost* 75 *francs;*

It is asked, what the rye, barley and wheat cost, each per measure?

Let x be the price of a measure of rye,

y, that of a measure of barley,

z, that of a measure of wheat.

To fulfil the first condition, we observe, that

30 measures of rye are worth $30x$,

20 measures of barley are worth $20y$,

10 measures of wheat are worth $10z$;

and as the whole must make 230 francs, we have the equation

$$30x + 20y + 10z = 230.$$

For the second condition, we have

15	measures of rye, worth	$15x$,
6	barley	$6y$,
12	wheat	$12z$,

and consequently,

$$15x + 6y + 12z = 138.$$

For the third condition, we have

10	measures of rye worth	$10x$,
5	barley	$5y$,
4	wheat	$4z$,

and consequently,

$$10x + 5y + 4z = 75.$$

The proposed question then will be brought into three equations;

$$30x + 20y + 10z = 230,$$
$$15x + 6y + 12z = 138,$$
$$10x + 5y + 4z = 75.$$

Before proceeding to the resolution, I examine the equations, to see if it is not possible to simplify them by dividing the two members of some one of them by the same number (12), and I find that the two members of the first may be divided by 10, and those of the second by 3. Having performed these divisions, I have only to occupy myself with the equations

$$3x + 2y + z = 23,$$
$$5x + 2y + 4z = 46,$$
$$10x + 5y + 4z = 75.$$

As I can select any one of the unknown quantities in order to deduce its value, I take that of z in the first equation, because this unknown quantity having no coefficient, its value will be entire or without a divisor, as follows;

$$z = 23 - 3x - 2y.$$

This value being substituted for z in the second and third equations, they become

$$5x + 2y + 92 - 12x - 8y = 46,$$
$$10x + 5y + 92 - 12x - 8y = 75;$$

and reducing the first member of each, we find

$$92 - 7x - 6y = 46,$$
$$92 - 2x - 3y = 75.$$

To proceed with these equations, which contain only two unknown quantities, I take in the first the value of the unknown quantity y, and I obtain

$$y = \frac{92 - 46 - 7x}{6}, \quad \text{or} \quad y = \frac{46 - 7x}{6},$$

and by substituting this value in the second equation, it becomes

$$92 - 2x - 3 \times \frac{46 - 7x}{6} = 75.$$

The denominator, 6, may be made to disappear by the usual method, but observing that the denominator is divisible by 3, I can simplify the fraction by multiplying it by 3, agreeably to article 54 of Arithmetic. I have then

$$92 - 2x - \frac{46 - 7x}{2} = 75.$$

The denominator 2 being made to disappear, it becomes

$$184 - 4x - 46 + 7x = 150;$$

the first member being reduced, gives

$$138 + 3x = 150,$$

whence

$$x = \frac{150 - 138}{3} = \frac{12}{3}, \quad \text{or} \quad x = 4.$$

Substituting this value in the expression for that of y, I find

$$y = \frac{46 - 7 \times 4}{6} = \frac{46 - 28}{6} = \frac{18}{6}, \quad \text{or} \quad y = 3;$$

and by substituting these values in the expression for that of z, we obtain

$$z = 23 - 3 \times 4 - 2 \times 3 = 23 - 12 - 6, \text{ or } z = 5.$$

It appears then, that the price of the rye per measure was 4 fr.,
that of the barley 3,
that of the wheat 5.

This example, while it illustrates the method given in the preceding article, ought to be attended to, on account of the abbreviations of calculation, which are performed in it.

80. I proceed now to resolve the following problem.

A man, who undertook to transport some porcelain vases of three different sizes, contracted that he would pay as much for each vessel that he broke, as he received for those which he delivered safe.

He had committed to him two small vases, four of a middle size, and nine large ones; he broke the middle sized ones, delivered all the others safe, and received the sum of 28 francs.

There were afterwards committed to him seven small vases, three of the middle size, and five large ones; he rendered this time the small and the middle sized ones, but broke the five large ones, and he received only 3 francs.

Lastly, he took charge of nine small vases, ten middle sized ones, and eleven large ones; all these last he broke, and received in consequence only 4 francs.

It is asked what was paid him for carrying a vase of each size?

Let x be the sum paid for carrying a small vase,
y, that for carrying a middle sized one,
z, that for carrying a large one.

It is evident, that each sum which the porter received, is the difference between what was due to him for the vessels delivered safe, and what he had to pay for those which were broken; accordingly, the three conditions of the problem furnish respectively the following equations;

$$\begin{aligned} 2x - 4y + 9z &= 28, \\ 7x + 3y - 5z &= 3, \\ 9x + 10y - 11z &= 4. \end{aligned}$$

The first of these equations gives

$$x = \frac{28 + 4y - 9z}{2};$$

and by substituting this value, the second and third equations become

$$\frac{196 + 28y - 63z}{2} + 3y - 5z = 3,$$

$$\frac{252 + 36y - 81z}{2} + 10y - 11z = 4.$$

Making the denominators to disappear, we have

$$\begin{aligned} 196 + 28y - 63z + 6y - 10z &= 6, \\ 252 + 36y - 81z + 20y - 22z &= 8; \end{aligned}$$

reducing the first member of each, we obtain

$$196 + 34y - 73z = 6,$$
$$252 + 56y - 103z = 8;$$

taking the value of y in the first of these equations, we find

$$y = \frac{73z - 190}{34}.$$

By means of this value, the second equation becomes

$$252 + 56 \times \frac{73z - 190}{34} - 103z = 8;$$

being cleared of the denominator 34, it is changed into

$$34 \times 252 + 56 \times 73z - 56 \times 190 - 34 \times 103z = 34 \times 8,$$

or into

$$8568 + 4088z - 10640 - 3502z = 272.$$

The reduction of the first member of this result, gives

$$586z - 2072 = 272,$$

whence we deduce

$$z = \frac{2344}{586}, \quad \text{or} \quad z = 4.$$

By going back with the value of z to that of y, we have

$$y = \frac{73 \times 4 - 190}{34} = \frac{292 - 190}{34} = \frac{102}{34}, \quad \text{or} \quad y = 3;$$

and with these two values, we find

$$x = \frac{28 + 4 \times 3 - 9 \times 4}{2} = \frac{28 + 12 - 36}{2} = \frac{4}{2}, \quad \text{or} \quad x = 2.$$

The prices then were 2 fr. for carrying a small vase,
3 one of a middle size,
4 a large one.

This example is sufficient to show how to proceed in all similar cases.

81. It sometimes happens, that all the unknown quantities do not enter at the same time into all the equations; the method, however, is not changed by this circumstance; it is sufficient, carefully to examine the connexion of the unknown quantities, in order to pass from one to the others.

Let there be, for example, the four equations,

$$3u - 2y = 2,$$
$$2x + 3y = 39,$$
$$5x - 7z = 11,$$
$$4y + 3z = 41,$$

containing the unknown quantities, u, x, y, and z.

With a little attention we see, that by taking the value of x

in the second equation, and substituting it in the third, the result containing only y and z, will, by being combined with the fourth equation, give the values of these two quantities; and having the value of y, we obtain those of u and x, by means of the first and second equations. The following is the process;

$$x = \frac{39 - 3y}{2}$$

$$5 \times \frac{39 - 3y}{2} - 7z = 11,$$

or
$$195 - 15y - 14z = 22,$$

or
$$15y + 14z = 173 \qquad (57).$$

The two equations

$$15y + 14z = 173,$$
$$4y + 3z = 41,$$

being resolved, give

$$y = 5, \quad z = 7;$$

and by means of these values, we have

$$x = \frac{39 - 3 \times 5}{2} = \frac{39 - 15}{2} = \frac{24}{2}, \quad \text{or} \quad x = 12,$$

$$u = \frac{2 + 2y}{3} = \frac{2 + 10}{3} = \frac{12}{3}, \quad \text{or} \quad u = 4.$$

The numbers sought then are

4, 12, 5, and 7.

82. The method now explained is applicable to literal equations, as well as to numerical ones; but the multitude of letters, which it is necessary to employ to represent generally the things given, when the number of equations and unknown quantities exceeds two, has led algebraists to seek for a more simple manner of expressing them. I shall treat of this in the following article; but in order to furnish the reader with the means of exercising himself in putting a problem into an equation, and resolving it, I have subjoined a number of questions, and have placed at the end of each the answer that is required.

1. *A father, being asked the age of his son, said, if from double the age that he is of now, you subtract triple of what he was six years ago, you have his present age.*

Answer. *The child was* 9 *years old.*

2. *Diophantus, the author of the most ancient book on Algebra, that has come down to us, passed a sixth part of his life in infancy, a twelfth part of it in youth; afterward he was married and passed*

in this state a seventh part, and five years more, when he had a son, whom he survived four years, and who attained only to half the age of his father, what was the age of Diophantus when he died?

Answer, 84 *years.*

3. *A merchant drew, every year, upon the stock he had in trade, the sum of* 1000 *francs for the expense of his family; still his property increased every year, by a third part of what remained after this deduction, and at the end of three years it was doubled; how much had he at the beginning of the first year?*

Answer, 14800 *francs.*

4. *A merchant has two kinds of tea, the first at* 14 *francs a pound, the second at* 18 *francs; how much ought he to take of each to make up a chest of* 100 *pounds, which should be worth* 1680 *francs?*

Answer, 30 *pounds of the first and* 70 *of the second.*

5. *A person filled, in* 12 *minutes, a vessel containing* 39 *gallons, with water, by means of two fountains, which were made to run in succession, and one discharged* 4 *gallons per minute and the other* 3, *how long did each fountain run?*

Answer, *the first* 3 *minutes, and the second* 9.

6. *At noon the hour and minute hands of a watch are together, at what point of the dial will they next be in conjunction?*

Answer, *at* 1 *hour* 5 *minutes and* $\frac{5}{11}$.

Obs. This problem refers itself to that of art. 65.

7. *A man, meeting some beggars, wishes to give them* 25 *cents each, but finds upon counting his money, that he wants* 10 *cents in order to do it; he then gives them only* 20 *cents each, and has* 25 *cents left; how much money had he, and what was the number of beggars?*

Answer, *he had* \$1,65, *and the number of beggars was* 7.

8. *Three brothers purchased an estate for* 50000 *francs, and the the first wanted, in order to complete the whole payment, half of the property of the second; the second would have paid the entire sum with the help of a third of what the first owned, and the third required, to make the same payment, in addition to what he had, a fourth part of what the first possessed; what was the amount of each one's property?*

Answer, *the first had* 30000 *francs, the second* 40000, *and the third* 42500.

9. *Three players after a game count their money, one had lost, the other two had gained each as much as he had brought to the play; after the second game, one of the players, who had gained before, lost*

and the two others gained each a sum equal to what he had at the beginning of this second game; at the third game, the player, who had gained till now, lost with each of the others a sum equal to that, which each of them possessed at the beginning of this last game; they then separated, each having 120 francs; how much had they each, when they commenced playing?

Answer, *he who lost at the first game, had* 195 *francs,*
he who lost at the second 105,
he who lost at the third 60.

General formulas for the resolution of Equations of the First Degree.

83. To obviate the inconvenience referred to in the beginning of the last article, we shall represent all the coefficients of the same unknown quantity by the same letter, but distinguish them by one or more accents, according to the number of equations.

General equations with two unknown quantities are written thus;

$$ax + by = c,$$
$$a'x + b'y = c'.$$

The coefficients of the unknown quantity x are both represented by a, those of y by b; but from the accent, which is placed over the letters in the second equation, it may be seen, that they are not considered as having the same value, as the corresponding ones in the first. Thus a' is a quantity different from a, b' a quantity different from b.

If there are three equations, they are expressed thus;

$$a\ x + b\ y + c\ z = d,$$
$$a'\ x + b'\ y + c'\ z = d',$$
$$a''x + b''y + c''z = d''.$$

All the coefficients of the unknown quantity x are designated by the letter a, those of y by b, those of z by c; but the several letters are distinguished by different accents, which show, that they denote different quantities. Thus a, a' a'', are three different quantities. The same may be said of b, b', b'', &c.

Following this method, if we have four unknown quantities, and four equations, we may write them thus;

$$a\ x + b\ y + c\ z + d\ u = e,$$
$$a'\ x + b'\ y + c'\ z + d'\ u = e',$$
$$a''\ x + b''\ y + c''\ z + d''\ u = e'',$$
$$a'''x + b'''y + c'''z + d'''u = e'''.$$

84. To avoid fractions, and simplify the calculation, we may vary the process of elimination in the following manner.

Let there be the equations

$$a x + b y = c,$$
$$a' x + b' y = c',$$

it is evident, that if one of the unknown quantities, x, for example, has the same coefficient in the two equations, we have only to subtract one of these equations from the other, in order to make this unknown quantity disappear. This may be seen at once in the equations

$$10 x + 11 y = 27,$$
$$10 x + 9 y = 15,$$

which give

$$11 y - 9 y = 27 - 15, \text{ or } 2 y = 12, \text{ or } y = 6.$$

It is evident, that the coefficients of x may be immediately made equal in the equations

$$a x + b y = c,$$
$$a' x + b' y = c',$$

by multiplying the two members of the first by a', the coefficient of x in the second, and the two members of the second by a, the coefficient of x in the first; we thus obtain,

$$a a' x + a' b y = a' c,$$
$$a a' x + a b' y = a c'.$$

Then subtracting the first of these from the second, the unknown quantity x disappears; and we have

$$(a b' - a' b) y = a c' - a' c,$$

an equation, which contains only the unknown quantity y; from this we may deduce,

$$y = \frac{a c' - c a'}{a b' - b a'}.$$

The method, we have just employed, may always be applied to equations of the first degree, to exterminate any one of the unknown quantities.

By exterminating, in the same manner, the unknown quantity y, we may find the value of x.

If we apply this process to three equations, containing x, y, and z, we may first exterminate x from the first and second, then from the first and third; we thus obtain two equations, which contain only y and z, from which we may exterminate y.

When this calculation is performed, the equation containing z,

to which we arrive, will have a factor common to all its terms, and consequently will not be the most simple, which may be obtained.

85. Bézout has given a very simple method for exterminating at once all the unknown quantities except one, and for reducing the question immediately to equations, which contain one unknown quantity less, than the equations proposed. Although this process is necessary, only when equations with three unknown quantities are employed, we shall, in order to give a complete view of the subject, begin by applying it to those, which contain only two.

Let there be the equations

$$ax + by = c,$$
$$a'x + b'y = c';$$

multiplying the first by any indeterminate quantity m, we have

$$amx + bmy = mc;$$

subtracting from this result the equation

$$a'x + b'y = c',$$

there remains

$$amx - a'x + bmy - b'y = cm - c',$$

or $$(am - a')x + (bm - b')y = cm - c'.$$

Since m is an indeterminate quantity, we may suppose it to be such, that $bm = b'$. In this case, the term multiplied by y disappears, and we have

$$x = \frac{cm - c'}{am - a'};$$

but since $bm = b'$, it follows that,

$$m = \frac{b'}{b};$$

therefore

$$x = \frac{\frac{cb'}{b} - c'}{\frac{ab'}{b} - a'} = \frac{cb' - bc'}{ab' - ba'}.$$

If, instead of supposing $bm = b'$, we make $am = a'$, the term, which contains x, will vanish, and we shall have

$$y = \frac{cm - c'}{bm - b'}.$$

the value of m will not be the same as before; for we shall have

$$m = \frac{a'}{a};$$

and by substituting this in the expression for y, we find

$$y = \frac{ca' - ac'}{ba' - ab'}.$$

If we change the signs of the numerator and denominator of this value of y, the denominator will become the same, as that in the expression for x, since we shall have

$$y = \frac{ac' - ca'}{ab' - ba'}.$$

86. Next let there be the three equations

$$\begin{aligned} a\ x + b\ y + c\ z &= d, \\ a'\ x + b'\ y + c'\ z &= d', \\ a''x + b''y + c''z &= d''; \end{aligned}$$

we shall be led, by an obvious analogy, to multiply the first of these equations by m, and the second by n, m and n being indeterminate quantities, to add together the results, and from the sum to subtract the third; by this means, all the equations will be employed at the same time, and the two new quantities m and n, which we may dispose of, as we please, will admit of any determinate value, which may be necessary to make both the unknown quantities disappear in the result. Having proceeded in this manner, and united the terms by which the same unknown quantity is multiplied, we shall have

$$\begin{aligned}(am + a'n - a'')\,x + (bm + b'n - b'')\,y + (cm + c'n - c'')\,z \\ = dm + d'n - d''.\end{aligned}$$

If we would make the unknown quantities x and y disappear, we must take the equations

$$\begin{aligned} am + a'n &= a'', \\ bm + b'n &= b'', \end{aligned}$$

and then we obtain

$$z = \frac{dm + d'n - d''}{cm + c'n - c''}.$$

From the two equations, in which m and n are the unknown quantities, it is easy to deduce the value of these quantities, by means of the results obtained in the preceding article; for it is only necessary to change in these results x into m, y into n, and to write instead of the letters

$$\left.\begin{matrix} a, & b, & c, \\ a', & b', & c', \end{matrix}\right\} \text{ the letters } \left\{\begin{matrix} a, & a', & a'', \\ b, & b', & b'', \end{matrix}\right.$$

which gives

$$m = \frac{a'' b' - b'' a'}{a\ b' - b\ a'},$$

$$n = \frac{a\ b'' - b\ a''}{a\ b' - b\ a'},$$

Substituting these values in the expression for z, and reducing all the terms to the same denominator, we have,†

$$z = \frac{d\,(b'\,a'' - a'\,b'') + d'\,(a\,b'' - b\,a'') - d''\,(a\,b' - b\,a')}{c\,(b'\,a'' - a'\,b'') + c'\,(a\,b'' - b\,a'') - c''\,(a\,b' - b\,a')}.$$

If we had made the terms containing x and z to disappear, we should have had y; the letters m and n would have depended upon the equations

$$a\,m + a'\,n = a'', \qquad c\,m + c'\,n = c'',$$

and proceeding as before, we should have found

$$y = \frac{d\,(c'\,a'' - a'\,c'') + d'\,(a\,c'' - c\,a'') - d''\,(a\,c' - c\,a')}{b\,(c'\,a'' - a'\,c'') + b'\,(a\,c'' - c\,a'') - b''\,(a\,c' - c\,a')}.$$

Lastly, by assuming the equations

$$b\,m + b'\,n = b'', \qquad c\,m + c'\,n = c'',$$

we make the terms multiplied by y and z to disappear; and we have

$$x = \frac{d\,(c'\,b'' - b'\,c'') + d'\,(b\,c'' - c\,b'') - d''\,(b\,c' - c\,b')}{a\,(c'\,b'' - b'\,c'') + a'\,(b\,c'' - c\,b'') - a''\,(b\,c' - c\,b')}.$$

These values being developed in such a manner, as to make the terms alternately positive and negative, if we change, at the same time, the signs of the numerator and denominator, in the first and third, we shall give them the following forms;

$$z = \frac{a\,b'\,d'' - a\,d'\,b'' + d\,a'\,b'' - b\,a'\,d'' + b\,d'\,a'' - d\,b'\,a''}{a\,b'\,c'' - a\,c'\,b'' + c\,a'\,b'' - b\,a'\,c'' + b\,c'\,a'' - c\,b'\,a''},$$

$$y = \frac{a\,d'\,c'' - a\,c'\,d'' + c\,a'\,d'' - d\,a'\,c'' + d\,c'\,a'' - c\,d'\,a''}{a\,b'\,c'' - a\,c'\,b'' + c\,a'\,b'' - b\,a'\,c'' + b\,c'\,a'' - c\,b'\,a''},$$

$$x = \frac{d\,b'\,c'' - d\,c'\,b'' + c\,d'\,b'' - b\,d'\,c'' + b\,c'\,d'' - c\,b'\,d''}{a\,b'\,c'' - a\,c'\,b'' + c\,a'\,b'' - b\,a'\,c'' + b\,c'\,a'' - c\,b'\,a''}.$$

87. Let there be the four equations

$$\begin{aligned} a\ x + b\ y + c\ z + d\ u &= e, \\ a'\ x + b'\ y + c'\ z + d'\ u &= e', \\ a''\,x + b''\,y + c''\,z + d''\,u &= e'', \\ a'''x + b'''y + c'''z + d'''u &= e'''; \end{aligned}$$

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† $$z = \frac{d\,\dfrac{a''\,b' - b''\,a'}{a\ b' - b\ a'} + d'\,\dfrac{a\,b'' - b\,a''}{a\,b' - b\,a'} - d''\,\dfrac{a\,b' - b\,a'}{a\,b' - b\,a'}}{c\,\dfrac{a''\,b' - b''\,a'}{a\ b' - b\ a'} + c'\,\dfrac{a\,b'' - b\,a''}{a\,b' - b\,a'} - c''\,\dfrac{a\,b' - b\,a'}{a\,b' - b\,a'}}.$$

if we multiply the first by m, the second by n, the third by p, and from the sum of their products subtract the fourth, we shall have

$$(am + a'n + a''p - a''')x + (bm + b'n + b''p - b''')y$$
$$+ (cm + c'n + c''p - c''')z + (dm + d'n + d''p - d''')u$$
$$= em + e'n + e''p - e'''.$$

In order to obtain u, we make

$$am + a'n + a''p = a''',$$
$$bm + b'n + b''p = b''',$$
$$cm + c'n + c''p = c''',$$

we then have

$$u = \frac{em + e'n + e''p - e'''}{dm + d'n + d''p - d'''}.$$

The preceding equations, which must give m, n, and p, may be resolved by means of the formulas found for the case of three unknown quantities. This method will appear very simple and convenient; but the nature of the results obtained above will furnish us with a rule for finding them without any calculation.

88. To begin with the most simple case, we take an equation with one unknown quantity, $ax = b$; from this we find

$$x = \frac{b}{a},$$

in which the numerator is the whole known term b, and the denominator the coefficient a, of the unknown quantity.

From the two equations

$$ax + by = c, \qquad a'x + b'y = c',$$

we have already deduced

$$x = \frac{cb' - bc'}{ab' - ba'}, \qquad y = \frac{ac' - ca'}{ab' - ba'}.$$

The denominator in this case also is composed of the letters a, a', b, b', by which the unknown quantities are multiplied. We first write a by the side of b, which gives ab; we then change the order of a and b, and obtain ba; prefixing to this the sign $-$ we have $ab - ba$; lastly, we place an accent over the last letter in each term, and the expression becomes $ab' - ba'$ for the denominator.

From this expression we may find the numerator. To obtain that for x, we have only to change each a into c, and each b into c for that of y, putting an accent over the last letter as before; in this way we find $cb' - bc'$ for the one, and $ac' - ca'$ for the

other. *The numerator may, therefore, be found from the denominator,* as well in cases where there are two unknown quantities, as when there is only one, *by changing the coefficient of the unknown quantity sought, into the known term or second member, and retaining the accents, which belonged to the coefficients.*

The same rule may be applied to equations with three unknown quantities, as we shall see by merely inspecting the values, which result from these equations. With respect to the denominator, it is necessary further to illustrate the method by which it is formed. Now, since in the case of two unknown quantities, the denominator presents all the possible transpositions of the letters a and b, by which the unknown quantities are multiplied, it may be supposed, that when there are three unknown quantities, their denominator will contain all the arrangements of the three letters a, b, c. These arrangements may be formed in the following manner.

We first make the transpositions $ab - ba$ with the two letters a and b, then, after the first term ab, write the third letter c, which gives abc; making this letter pass through all the places, observing each time to change the sign, and not to derange the order in which a and b respectively stand, we obtain

$$abc - acb + cab.$$

Proceeding in the same manner with respect to the second term $-ba$, we find

$$-bac + bca - cba;$$

connecting these products with the preceding, and placing over the second letter one accent, and over the third two, we have

$$ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a'',$$

a result, which agrees with that presented by the formulas, obtained above.

From this it is obvious, that, in order to form a denominator in the case of four unknown quantities, it is necessary to introduce the letter d into each of the six products,

$$abc - acb + cab - bac + bca - cba,$$

and to make it occupy successively all the places. The term abc, for example, will give the four following;

$$abcd - abdc + adbc - dabc.$$

If we observe the same method in regard to the five other products, the whole result will be twenty-four terms, in each of

which, the second letter will have one accent, the third two, and the fourth three. The numerators of the unknown quantities u, z, y, and x, are found by the rule already given.*

89. We may employ these formulas for the resolution of numerical equations. In doing this, we must compare the terms of the equations proposed with the corresponding terms of the general equations, given in the preceding articles.

To resolve, for example, the three equations

$$\begin{aligned} 7x + 5y + 2z &= 79, \\ 8x + 7y + 9z &= 122, \\ x + 4y + 5z &= 55, \end{aligned}$$

it is necessary to compare the terms with those of the equations given in art. 86. We have then

$$\begin{aligned} a &= 7,\ b = 5,\ c = 2,\ d = 79, \\ a' &= 8,\ b' = 7,\ c' = 9,\ d' = 122, \\ a'' &= 1,\ b'' = 4,\ c'' = 5,\ d'' = 55. \end{aligned}$$

Substituting these values in the general expressions for the unknown quantities x, y, and z, and going through the operations, which are indicated, we find

$$x = 4, \qquad y = 9, \qquad z = 3.$$

It is important to remark, that the same expressions may be employed, even when the proposed equations are not, in all their terms, affected with the sign $+$, as the general equations, from which these expressions are deduced appear to require. If we have, for example,

$$\begin{aligned} 3x - 9y + 8z &= 41, \\ -5x + 4y + 2z &= -20, \\ 11x - 7y - 6z &= 37, \end{aligned}$$

in comparing the terms of these equations with the corresponding ones in the general equations, we must attend to the signs, and the result will be

$$\begin{aligned} a &= +3,\ b = -9,\ c = +8,\ d = +41, \\ a' &= -5,\ b' = +4,\ c' = +2,\ d' = -20, \\ a'' &= +11,\ b'' = -7,\ c'' = -6,\ d'' = +37. \end{aligned}$$

We are then to determine by the rules given in art. 31, the sign,

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* M. Laplace, in the second part of the Mémoires de l'Académie des Sciences for 1772, p. 294, has demonstrated these rules *à priori*. See also *Annales des Mathématiques pures appliquées*, by M. Gergonne, vol. iv, p. 148.

which each term of the general expressions for x, y, and z, ought to have, according to the signs of the factors of which it is composed. Thus we find, for example, that the first term of the common denominator, which is $a\,b'\,c''$, becoming $+ 3 \times + 4 \times - 6$, changes the sign of the product, and gives $- 72$. If we observe the same method with respect to the other terms, both of the numerators and denominators, taking the sum of those, which are positive, and also of those which are negative, we obtain

$$x = \frac{2774 - 2834}{592 - 622} = \frac{-60}{-30} = +2,$$

$$y = \frac{3022 - 2932}{592 - 622} = \frac{+90}{-30} = -3,$$

$$z = \frac{3859 - 3889}{592 - 622} = \frac{-30}{-30} = +1.$$

Equations of the Second Degree, having only one unknown Quantity.

90. Hitherto I have been employed upon equations of the *first degree*, or such as involve only the first power of the unknown quantities; but were the question proposed, *To find a number, which, multiplied by five times itself, will give a product equal to* 125; if we designate this number by x, five times the same will be $5\,x$, and we shall have

$$5\,x^2 = 125.$$

This is an equation of the *second degree*, because it contains x^2, or the second power of the unknown quantity. If we free this second power from its coefficient 5, we obtain

$$x^2 = \frac{125}{5}, \qquad \text{or} \qquad x^2 = 25.$$

We cannot here obtain the value of the unknown quantity x, as in art. 11, and the question amounts simply to this, to find a number which, multiplied by itself, will give 25. It is obvious that this number is 5; but it seldom happens that the solution is so easy; hence arises this new numerical question; *to find a number, which, multiplied by itself, will give a product equal to a proposed number;* or, which is the same thing, from the second power of a number, to retrace our steps to the number from which it is derived, and which is called the *square root.* I shall proceed, in the first place, to resolve this question, as it is involved in the determination of the unknown quantities, in all equations of the second degree.

91. The method employed in finding or *extracting* the roots of numbers, supposes the second power of such, as are expressed by only one figure to be known. See the nine primitive numbers with their second powers written under them respectively.

1	2	3	4	5	6	7	8	9
1	4	9	16	25	36	49	64	81.

It is evident from this table, that the second power of a number expressed by one figure, contains only two figures; 10, which is the least number expressed by two figures, has for its square a number composed of three, 100. In order to resolve the second power of a number consisting of two figures, we must attend to the method by which it is formed; for this purpose we must inquire, how each part of the number 47, for example, is employed in the production of the square of this number.

We may resolve 47 into 40 + 7, or into 4 tens and 7 units; if we represent the tens of the proposed number by a, and the units by b, the second power will be expressed by

$$(a + b)(a + b) = a^2 + 2ab + b^2;$$

that is, it is made up of three parts, namely, *the square of the tens, twice the product of the tens multiplied by the units, and the square of the units.* In the example we have taken, $a = 4$ tens or 40 units, and $b = 7$; we have then

$$\begin{aligned} a^2 &= 1600 \\ 2ab &= 560 \\ b^2 &= 49 \\ \hline \end{aligned}$$

$$\text{Total,} \quad a^2 + 2ab + b^2 = 2209 = 47 \times 47.$$

Now in order to return, by a reverse process, from the number 2209 to its root, we may observe, that the square of the tens, 1600, has no figure, which denotes a rank inferior to hundreds, and that it is the greatest square, which the 22 hundreds, comprehended in 2209, contain; for 22 lies between 16 and 25, that is, between the square of 4 and that of 5, as 47 falls between 4 tens or 40, and 5 tens or 50.

We find, therefore, upon examination, that the greatest square contained in 22 is 16, the root of which 4 expresses the number of tens in the root of 2209; subtracting 16 hundreds, or 1600 from 2209, the remainder 609 contains double the product of the tens by the units, 560, and the square of the units 49. But as double the product of the tens by the units has no figure inferior

to tens, it must be found in the two first figures 60 of the remainder 609, which contain also the tens, arising from the square of the units. Now, if we divide 60 by double of the tens 8, and neglect the remainder, we have a quotient 7 equal to the units sought. If we multiply 8 by 7, we have double the product of the tens by the units, 560; subtracting this from the whole remainder 609, we obtain a difference 49, which must be, and in fact is, the square of the units.

This process may be exhibited thus;

22,09	47
16	87
60,9	
60 9	
000	

We write the proposed number in the manner of a dividend, and assign for the root the usual place of the divisor. We then separate the units and tens by a comma, and employ only the two first figures on the left, which contain the square of the tens found in the root. We seek the greatest square 16, contained in these two figures, put the root 4 in its assigned place, and subtract 16 from 22. To the remainder we bring down the two other figures, 09, of the proposed number, separating the last, which does not enter into double the product of the tens by the units, and divide the remainder on the left by 8, double the tens in the root, which gives for the quotient the units 7. In order to collect into one expression the two last parts of the square contained in 609, we write 7 by the side of 8, which gives 87, equal to double the tens plus the units, or $2a + b$; this multiplied by 7 or b, reproduces $609 = 2ab + b^2$, or double the product of the tens by the units, plus the square of the units. This being subtracted leaves no remainder, and the operation shows, that 47 is the square root of 2209.

If it were required to extract the square root of 324; the operation would be as follows;

3,24	18
1	
22,4	28
22 4	
000	

Proceeding as in the last example, we obtain 1 for the place of tens of the root; this doubled gives the number 2, by which the two first figures 22 of the remainder are to be divided. Now 22 contains 2 eleven times, but the root can neither be more than 10, nor 10; even 9 is in fact too large, for if we write 9 by the side of 2, and multiply 29 by 9, as the rule requires, the result is 261, which cannot be subtracted from 224. We are, therefore, to consider the division of 22 by 2, only as a means of approximating the units, and it becomes necessary to diminish the quotient obtained, until we arrive at a product, which does not exceed the remainder 224. The number 8 answers to this condition, since $8 \times 28 = 224$; therefore, the root sought is 18.

By resolving the square of 18 into its three parts, we find

$$\begin{array}{rl} a^2 = & 100 \\ 2ab = & 160 \\ b^2 = & 64 \\ \hline \text{Total,} & 324 = 18 \times 18, \end{array}$$

and it may be seen, that the 6 tens, contained in the square of the units, being united to 160, double the product of the tens by the units, alters this product in such a manner, that a division of it by double the tens will not give exactly the units.

92. It will not be difficult, after what has been said, to extract the square root of a number, consisting of three or four figures; but some further observations, founded upon the principles above laid down, may be necessary to enable the reader to extract the root of any number whatever.

No number less than 100 can have a square consisting of more than four figures, since that of 100 is 10000, or the least number expressed by five figures. In order, therefore, to analyze the square of any number exceeding 100, of 473, for example, we may resolve it into $470 + 3$, or 47 tens plus 3 units. To obtain its square from the formula,

$$a^2 + 2ab + b^2,$$

we make $a = 47$ tens $= 470$ units, $b = 3$ units, then

$$\begin{array}{rr} a^2 = & 220900 \\ 2ab = & 2820 \\ b^2 = & 9 \\ \hline \text{Total,} & 223729 = 473 \times 473. \end{array}$$

In this example, it is evident that the square of the tens has no

figure inferior to hundreds, and this is a general principle, since tens multiplied by tens, always give hundreds, (*Arith.* 32).

It is therefore in the part 2237, which remains on the left of the proposed number, after we have separated the tens and units, that it is necessary to seek the square of the tens; and as 473 lies between 47 tens, or 470, and 48 tens, or 480, 2237 must fall between the square of 47 and that of 48; hence the greatest square contained in 2237, will be the square of 47, or that of the tens of the root. In order to find these tens, we must evidently proceed, as if we had to extract the square root of 2237 only; but instead of arriving at an exact result, we have a remainder, which contains the hundreds arising from double the product of the 47 tens multiplied by the units.

The operation is as follows;

22,37,29	473
16	87
63,7	943
60 9	
282,9	
282 9	
0	

We first separate the two last figures 29, and in order to extract the root of the number 2237, which remains on the left, we further separate the two last figures 37 of this number; the proposed number is then divided into portions of two figures, beginning on the right and advancing to the left. Proceeding with the two first portions as in the preceding article, we find the two first figures 47 of the root; but we have a remainder 28, which, joined to the two figures 29 of the last portion, contains double the product of the 47 tens by the units, and the square of the units. We separate the figure 9, which forms no part of double the product of the tens by the units, and divide 282 by 94, double the 47 tens; writing the quotient 3 by the side of 94, and multiplying 943 by 3, we obtain 2829, a number exactly equal to the last remainder, and the operation is completed.

93. In order to show, by what method we are to proceed with any number of figures, however great, I shall extract the root of

22391824. Whatever this root may be, we may suppose it capable of being resolved into tens and units, as in the preceding examples. As the square of the tens has no figure inferior to hundreds, the two last figures 24 cannot make a part of it; we may therefore separate them, and the question will be reduced to this, to find the greatest square contained in the part 223918, which remains on the left. This part consisting of more than two figures, we may conclude, that the number, which expresses the tens in the root sought, will have more than one figure; it may therefore be resolved, like the others, into tens and units. As the square of the tens does not enter into the two last figures 18 of the number 223918, it must be sought in the figures 2239, which remain on the left; and since 2239 still consists of more than two figures, the square, which is contained in it must have a root, which consists of at least two; the number which expresses the tens sought will therefore have more than one figure; it is then, lastly, in 22 that we must seek the square of that, which represents the units of the highest place in the root required. By this process, which may be extended to any length we please, the proposed number may be divided into portions of two figures from right to left; it must be understood, however, that the last figure on the left may consist of only one figure.

Having divided the proposed number into portions as below, we proceed with the three first portions, as in the preceding article; and when we have found the three first figures 473 of the root, to the remainder 189, we bring down the fourth portion 24, and consider the number 18924, as containing double the product of the 473 tens already found by the units sought, plus the square of these units. We separate the last figure 4; divide those, which remain on the left, by 946, double of 473, and then make trial of the quotient 2, as in the preceding examples.

22,39,18,24	4732
16	87
63,9	943
60 9	9462
301,8	
282 9	
1892,4	
1892 4	
0000 0	

Here the operation, in the present case, terminates; but it is very obvious, that if we had one portion more, the four figures already found 4732, would express the tens of a root, the units of which would remain to be sought; we should proceed, there-

fore to divide the remainder now found, together with the first figure of the following portion, by double of these tens, and so on for each of the portions to be successively brought down.

94. If, after having brought down a portion, the remainder, joined to the first figure of this portion, does not contain double of the figures already found, a cypher must be placed in the root; for the root, in this case, will have no units of this rank; the following portion is then to be brought down, and the operation to be continued as before. The example subjoined will illustrate this case. The quantities to be subtracted are not put down, but the subtractions are supposed to be performed mentally, as in division.

49,42,09	703
04,20, 9	1403
0 00 0	

95. Every number, it will be perceived, is not a perfect square. If we look at the table given, page 100, we shall see that between the squares of each of the nine primitive numbers, there are intervals comprehending many numbers, which have no assignable root; 45, for instance, is not a square, since it falls between 36 and 49. It very often happens, therefore, that the number, the root of which is sought, does not admit of one; but if we attempt to find it, we obtain for the result the root of the greatest square, which the number contains. If we seek, for example, the root of 2276, we obtain 47, and have a remainder 67, which shows, that the greatest square contained in 2276, is that of 47 or 2209.

As a doubt may sometimes arise, after having obtained the root of a number, which is not a perfect square, whether the root found be that of the greatest square contained in the number, I shall give a rule, by which this may be readily determined. As the square of $a + b$ is

$$a^2 + 2ab + b^2,$$

if we make $b = 1$, the square of $a + 1$ will be

$$a^2 + 2a + 1,$$

a quantity which differs from a^2, the square of a, by double of a plus unity. Therefore, *if the root found can be augmented by unity, or more than unity, its square, subtracted from the proposed number, will leave a remainder at least equal to twice this root plus unity.* Whenever this is not the case, the root obtained will be, in fact, that of the greatest square contained in the number proposed.

96. Since a fraction is multiplied by another fraction, when their numerators are multiplied together, and their denominators

together, it is evident that the product of a fraction multiplied by itself, or *the square of a fraction is equal to the square of its numerator, divided by the square of its denominator.* Hence it follows, that to extract *the square root of a fraction, we extract the square root of its numerator and that of its denominator.* Thus the root of $\frac{25}{64}$ is $\frac{5}{8}$, because 5 is the square root of 25, and 8 that of 64.

It is very important to remark, that not only are the squares of fractions, properly so called, always fractions, but *every fractional number, which is irreducible,* (Arith. 59) *will, when multiplied by itself, give a fractional result, which is also irreducible.*

97. This proposition depends upon the following; *Every prime number* P, *which will divide the product* AB *of two numbers* A *and* B, *will necessarily divide one of these numbers.*

Let us suppose, that it will not divide B, and that B is the greater; if we designate the entire part of the quotient by q, and the remainder by B', we have

$$B = qP + B',$$

multiplying by A, we obtain

$$AB = qAP + AB',$$

and dividing the two members of this equation by P, we have

$$\frac{AB}{P} = qA + \frac{AB'}{P};$$

from which it appears, that if AB be divisible by P, the product AB' will be divisible by the same number. Now B', being the remainder after the division of B by P, must be less than P; therefore B' cannot be divided by P; if we divide P by B' we have a quotient q' and a remainder B''; if further we divide P by B'', we have a quotient q'' and a remainder B''', and so on, since P is a prime number.

We have, therefore, the following series of equations;

$$P = q'B' + B'', \quad P = q''B'' + B''', \text{ \&c.}$$

multiplying each of these by A, we obtain

$$AP = q'AB' + AB'', \quad AP = q''AB'' + AB''', \text{ \&c.}$$

dividing by P, we have

$$A = q'\frac{AB'}{P} + \frac{AB''}{P}, \quad A = q''\frac{AB''}{P} + \frac{AB'''}{P}, \text{ \&c.}$$

From these results it is evident, that if AB' be divisible by P, the products AB'', AB''', &c. will also be divisible by it. But the remainders B', B'', B''', &c. are becoming less and less, continually,

till they finally terminate in unity, for the operation exhibited above may be continued in the same manner, while the remainder is greater than 1, since P is a prime number. Now when the remainder becomes unity, we have the product $A \times 1$, which must be divisible by P; therefore A also must be divisible by P.

Hence, if the prime number P, which we have supposed not to divide B, will not divide A, it will not divide the product of these numbers.

(*This demonstration is taken principally from the* Théorie des nombres of M. Legendre.)

98. Now when the fraction $\frac{b}{a}$ is irreducible, there is no prime number, which will divide, at the same time, b and a; but, from the preceding demonstration, it is evident, that every prime number, which will not divide a, will not divide $a \times a$, or a^2, every prime number, which will not divide b, will not divide $b \times b$, or b^2; the numbers a^2 and b^2 are, therefore, in this case, prime to each other; and consequently the square $\frac{b^2}{a^2}$ of the fraction $\frac{b}{a}$, being irreducible, as well as as the fraction itself, cannot become an entire number [B].

99. From this last proposition it follows, that *entire numbers, except only such as are perfect squares, admit of no assignable root, either among whole numbers or fractions.* Yet it is evident, that there must be a quantity, which, multiplied by itself, will produce any number whatever, 2276, for instance, and that, in the present case, this quantity lies between 47 and 48; for 47×47 gives a product less than this number, and 48×48 gives one greater. Dividing then the difference between 47 and 48 by means of fractions, we may obtain numbers that, multiplied by themselves, will give products greater than the square of 47, but less than that of 48, and which will approach nearer and nearer to the number 2276.

The extraction of the square root, therefore, applied to numbers, which are not perfect squares, makes us acquainted with a new species of numbers, in the same manner, as division gives rise to fractions; but there is this difference between fractions and the roots of numbers, which are not perfect squares; that the former, which are always composed of a certain number of parts of unity, have with unity a *common measure*, or a relation

which may be expressed by whole numbers, which the latter have not.

If we conceive unity to be divided into five parts, for example, we express the quotient arising from the division of 9 by 5, or $\frac{9}{5}$, by nine of these parts; $\frac{1}{5}$ then, being contained five times in unity, and nine times in $\frac{9}{5}$, is the *common measure* of unity and the fraction $\frac{9}{5}$, and the relation these quantities have to each other is that of the entire numbers 5 and 9.

Since whole numbers, as well as fractions, have a common measure with unity, we say that these quantities are *commensurable* with unity, or simply that they are *commensurable;* and since their *relations* or *ratios*, with respect to unity, are expressed by entire numbers, we designate both whole numbers and fractions, by the common name of *rational numbers.*

On the contrary, the square root of a number, which is not a perfect square, is *incommensurable* or *irrational*, because, as it cannot be represented by any fraction, into whatever number of parts we suppose unity to be divided, no one of these parts will be sufficiently small to measure exactly, at the same time, both this root and unity.

In order to denote, in general, that a root is to be extracted, whether it can be exactly obtained or not, we employ the character $\sqrt{}$, which is called a *radical* sign;

$\sqrt{16}$ is equivalent to 4,

$\sqrt{2}$ is *incommensurable* or *irrational.*

100. Although we cannot obtain, either among whole numbers or fractions, the exact expression for $\sqrt{2}$, yet we may approximate it, to any degree we please, by converting this number into a fraction, the denominator of which is a perfect square. The root of the greatest square contained in the numerator will then be that of the proposed number expressed in parts, the value of which will be denoted by the root of the denominator.

If we convert, for example, the number 2 into twenty-fifths, we have $\frac{50}{25}$. As the root of 50 is 7, so far as it can be expressed in whole numbers, and the root of 25 exactly 5, we obtain $\frac{7}{5}$, or $1\frac{2}{5}$ for the root of 2, to within one fifth.

101. This process, founded upon what was laid down in article 96, that the square of a fraction is expressed by the square of the numerator divided by the square of the denominator, may evidently be applied to any kind of fraction whatever, and more

readily to decimals than to others. It is manifest, indeed, from the nature of multiplication, that the square of a number expressed by tenths will be hundredths, and that the square of a number expressed by hundredths will be ten thousandths, and so on; and consequently, that *the number of decimal figures in the square is always double that of the decimal figures in the root.* The truth of this remark is further evident from the rule observed in the multiplication of decimal numbers, which requires that a product should contain as many decimal figures, as there are in both the factors. In any assumed case, therefore, the proposed number, considered as the product of its root multiplied by itself, must have twice as many decimal figures as its root.

From what has been said, it is clear, that in order to obtain the square root of 227, for example, to within one hundredth, it is necessary to reduce this number to ten thousandths, that is, to annex to it four cyphers, which gives 2270000 ten thousandths. The root of this may be extracted in the same manner, as that of an equal number of units; but to show that the result is hundredths, we separate the two last figures on the right by a comma. We thus find that the root of 227 is 15,06, accurate to hundredths. The operation may be seen below;

2,27,00.00	1506
12,7	25
2 00 00	3006
19 64	

If there are decimals already in the proposed number, they should be made even. To extract, for example, the root of 51,7, we place one cypher after this number, which makes it hundredths; we then extract the root of 51,70. If we proposed to have one decimal more, we should place two additional cyphers after this number, which would give 51,7000; we should then obtain 7,19 for the root.

If it were required to find the square root of the numbers 2 and 3 to seven places of decimals, we should annex fourteen cyphers to these numbers; the result would be

$$\sqrt{2} = 1,4142136, \qquad \sqrt{3} = 1,7320508.$$

102. When we have found more than half the number of figures, of which we wish the root to consist, we may obtain the rest simply by division. Let us take, for example, 32976; the square root of this number is 181, and the remainder, 215. If

we divide this remainder 215, by 362, double of 181, and extend the quotient to two decimal places, we obtain 0,59, which must be added to 181; the result will be 181,59 for the root of 32976, which is accurate to within one hundredth.

In order to prove that this method is correct, let us designate the proposed number by N, the root of the greatest square contained in this number by a, and that which it is necessary to add to this root to make it the exact root of the proposed number by b; we have then

$$N = a^2 + 2ab + b^2,$$

from which we obtain

$$N - a^2 = 2ab + b^2;$$

dividing this by $2a$, we find

$$\frac{N - a^2}{2a} = b + \frac{b^2}{2a}.$$

From this result it is evident, that the first member may be taken for the value of b, so long as the quantity $\frac{b^2}{2a}$ is less than a unit of the lowest place found in b. But as the square of a number cannot contain more than twice as many figures as the number itself, it follows, that if the number of figures in a exceeds double those in b, the quantity $\frac{b^2}{2a}$ will then be a fraction.

In the preceding example, $a = 181$ units, or 18100 hundredths, and consequently contains one figure more than the square of 59 hundredths; the fraction then $\frac{b^2}{2a}$ becomes, in this case, $\frac{(59)^2}{2 \times 18100} = \frac{3481}{36200}$, and is less than a unit of the second part 59, or than a hundredth of a unit of the first.

103. This leads to a method of approximating the square root of a number by means of vulgar fractions. It is founded on the circumstance, that a, being the root of the greatest square contained in N, b is necessarily a fraction, and $\frac{b^2}{2a}$ being much smaller than b, may be neglected.

If it were required, for example, to extract the square root of 2; as the greatest square contained in this number is 1, if we subtract this, we have a remainder, 1. Dividing this remainder by double of the root, we obtain $\frac{1}{2}$; taking this quotient for the value of the quantity b, we have, for the first approximation to

the root, $1 + \frac{1}{2}$, or $\frac{3}{2}$. Raising this root to its square, we find $\frac{9}{4}$, which, subtracted from 2 or $\frac{8}{4}$, gives for a remainder $-\frac{1}{4}$. In this case the formula

$$\frac{N-a^2}{2a} = b + \frac{b^2}{2a},$$

becomes

$$-\frac{1}{12} = b + \frac{b^2}{2a}.$$

Substituting $-\frac{1}{12}$ for b, we have for the second approximation $\frac{3}{2} - \frac{1}{12} = \frac{17}{12}$; taking the square of $\frac{17}{12}$, we find $\frac{289}{144}$, a quantity, which still exceeds 2 or $\frac{288}{144}$. Substituting $\frac{17}{12}$ for a, we obtain

$$-\frac{1}{12 \times 34} = b + \frac{b^2}{2a};$$

which gives

$$b = -\frac{1}{12 \times 34} = -\frac{1}{408};$$

the third approximation will then be

$$\frac{17}{12} - \frac{1}{12 \times 34} = \frac{17 \times 34 - 1}{408} = \frac{577}{408}.$$

This operation may be easily continued to any extent we please. I shall give, in the *Supplement* to this treatise, other formulas more convenient for extracting roots in general.

104. In order to approximate the square root of a fraction, the method, which first presents itself, is, to extract, by approximation, the square root of the numerator and that of the denominator; but with a little attention it will be seen, that we may avoid one of these operations by making the denominator a perfect square. This is done by multiplying the two terms of the proposed fraction by the denominator. If it were required, for example, to extract the square root of $\frac{3}{7}$, we might change this fraction into

$$\frac{3 \times 7}{7 \times 7} = \frac{21}{49},$$

by multiplying its two terms by the denominator, 7. Taking the root of the greatest square contained in the numerator of this fraction, we have $\frac{4}{7}$ for the root of $\frac{3}{7}$, accurate to within $\frac{1}{7}$.

If a greater degree of exactness were required, the fraction $\frac{3}{7}$ must be changed by approximation or otherwise into another, the denominator of which is the square of a greater number than 7. We shall have, for example, the root sought within $\frac{1}{15}$, if we convert $\frac{3}{7}$ into 225ths, since 225 is the square of 15; thus the fraction becomes $\frac{675}{7}$ of one 225th, or $\frac{96}{225}$, within $\frac{1}{225}$; the root of

$\frac{96}{225}$ falls between $\frac{9}{15}$ and $\frac{10}{15}$, but approaches nearer to the second fraction than to the first, because 96 approaches nearer to a hundred than to 81; we have then $\frac{10}{15}$ or $\frac{2}{3}$ for the root of $\frac{3}{7}$ within $\frac{1}{15}$.

By employing decimals in approximating the root of the numerator of the fraction $\frac{21}{49}$, we obtain 4,583 for the approximate root of the numerator 21, which is to be divided by the root of the new denominator. The quotient thence arising, carried to three places of decimals, becomes 0,655.

105. We are now prepared to resolve all equations involving only the second power of the unknown quantity connected with known quantities.

We have only to *collect into one member all the terms containing this power, to free it from the quantities, by which it is multiplied* (11); *we then obtain the value of the unknown quantity by extracting the square root of each member.*

Let there be, for example, the equation

$$\frac{5}{7}x^2 - 8 = 4 - \frac{2}{3}x^2.$$

Making the divisors to disappear, we find first

$$15x^2 - 168 = 84 - 14x^2.$$

Transposing to the first member the term $14x^2$, and to the second the term 168, we have

$$15x^2 + 14x^2 = 84 + 168,$$

or

$$29x^2 = 252,$$

and

$$x^2 = \frac{252}{29},$$

$$x = \sqrt{\frac{252}{29}}.$$

It should be carefully observed, that to denote the root of the fraction $\frac{252}{29}$, the sign $\sqrt{\ }$ is made to descend below the line, which separates the numerator from the denominator. If it were written thus, $\frac{\sqrt{252}}{29}$, the expression would designate the quotient arising from the square root of the number 252 divided by 29; a result different from $\sqrt{\frac{252}{29}}$, which denotes, that the division is to be performed before the root is extracted.

Let there be the literal equation

$$ax^2 + b^3 = cx^2 + d^3;$$

proceeding as with the above, we obtain successively

$$ax^2 - cx^2 = d^3 - b^3,$$

$$x^2 = \frac{d^3 - b^3}{a - c},$$

$$x = \sqrt{\frac{d^3 - b^3}{a - c}}.$$

I would remark here, that in order to designate the square root of a compound quantity, the upper line must be extended over the whole radical quantity.

The root of the quantity $4a^2b - 2b^3 + c^3$ is written thus,

$$\sqrt{4a^2b - 2b^3 + c^3},$$

or rather

$$\sqrt{}(4a^2b - 2b^3 + c^3),$$

by substituting, for the line extended over the radical quantity, a parenthesis including all the parts of the quantity, the root of which is required. This last expression may often appear preferable to the other (35).

In general, every equation of the second degree of the kind we are here considering, may, by a transposition of its terms, be reduced to the form

$$\frac{px^2}{q} = a,$$

$\frac{p}{q}$ designating the coefficient, whatever it may be, of x^2. We then obtain

$$x^2 = \frac{aq}{p},$$

$$x = \sqrt{\frac{aq}{p}}.$$

106. With respect to numbers taken independently, this solution is complete, since it is reduced to an operation upon the number either entire or fractional, which the quantity $\frac{aq}{p}$ represents, an arithmetical operation leading always to an exact result, or to one, which approaches the truth very nearly. But in regard to the signs, with which the quantities may be affected, there remains, after the square root is extracted, an ambiguity, in consequence of which every equation of the second degree admits of two solutions, while those of the first degree admit of only one.

Thus in the general equation $x^2 = 25$, the value of x, being the quantity, which, raised to its square, will produce 25, may, if we consider the quantities algebraically, be affected either with the sign $+$ or $-$; for whether we take $+5$, or -5, for this value, we have for the square

$$+5 \times +5 = +25, \quad \text{or} \quad -5 \times -5 = +25;$$

we may therefore take

$$x = + 5,$$

or $$x = - 5.$$

For the same reason, from the general equation

$$x^2 = \frac{aq}{p},$$

we have

$$x = + \sqrt{\frac{aq}{p}},$$

or

$$x = - \sqrt{\frac{aq}{p}}.$$

Both these expressions are comprehended in the following;

$$x = \pm \sqrt{\frac{aq}{p}},$$

in which the double sign $\pm$ shows, that the numerical value of

$$\sqrt{\frac{aq}{p}},$$

may be affected with the sign $+$ or $-$.

From what has been said, we deduce the general rule, *that the double sign $\pm$ is to be considered as affecting the square root of every quantity whatever.*

It may be here asked, why x, as it is the square root of x^2, is not also affected with the double sign $\pm$? We may answer, first, that the letter x, having been taken without a sign, that is, with the sign $+$, as the representative of the unknown quantity, it is its value when in this state, which is the subject of inquiry; and, that when we seek a number x, the square of which is b, for example, there can be only two possible solutions; $x = + \sqrt{b}$, $x = - \sqrt{b}$. Again, if in resolving the equation $x^2 = b$, we write $\pm x = \pm \sqrt{b}$, and arrange these expressions in all the different ways, of which they are capable, namely,

$$+ x = + \sqrt{b}, \quad - x = - \sqrt{b},$$
$$+ x = - \sqrt{b}, \quad - x = + \sqrt{b},$$

we come to no new result, since by transposing all the terms of the equations $- x = - \sqrt{b}$, $- x = + \sqrt{b}$, or which is the same thing, by changing all the signs (57), these equations become identical with the first.

107. It follows from the nature of the signs, that if the second member of the general equation

$$x^2 = \frac{aq}{p}$$

were a negative number, the equation would be absurd, since the square of a quantity affected either with the sign $+$, or $-$, having always the sign $+$, no quantity, the square of which is negative, can be found either among positive or negative quantities.

This is what is to be understood, when we say, that *the root of a negative quantity is imaginary.*

If we were to meet with the equation

$$x^2 + 25 = 9,$$

we might deduce from it

$$x^2 = 9 - 25,$$

or
$$x^2 = \quad -16;$$

but, there is no number, which, multiplied by itself, will produce -16. It is true, that -4 multiplied by $+4$, gives -16; but as these two quantities have different signs, they cannot be considered as equal, and consequently their product is not a square. This species of contradiction, which will be more fully considered hereafter, must be carefully distinguished from that mentioned in art. 58, which disappears by simply changing the sign of the unknown quantity; here it is the sign of the square x^2, which is to be changed.

108. To be complete, an equation of the second degree, with only one unknown quantity, must have three kinds of terms, namely, those involving the square of the unknown quantity, others containing the unknown quantity of the first degree, lastly, such as comprehend only known quantities. The following equations are of this kind;

$$x^2 - 4x = 12, \qquad 4x - \tfrac{3}{5}x^2 = 4 - 2x.$$

The first is, in some respects, more simple than the second, because it contains only three terms, and the square of x is positive, and has only unity for a coefficient. It is to this last form, that we are always to reduce equations of the second degree, before resolving them; they may then be represented by this formula,
$$x^2 + px = q,$$
in which p and q denote known quantities, either positive or negative.

It is evident, that we may reduce all equations of the second degree to this state, 1. by collecting into one member all the terms involving x (10), 2. by changing the sign of each term of the equation, in order to render that of x^2 positive, if it was before negative (57), 3. by dividing all the terms of the equation

by the multiplier of x^2, if this square have a multiplier (11), or by multiplying by its divisor, if it be divided by any number (12).

If we apply what has just been said to the equation

$$4x - \tfrac{3}{5}x^2 = 4 - 2x,$$

we have, by collecting into the first member all the terms involving x,

$$-\tfrac{3}{5}x^2 + 6x = 4,$$

by changing the signs,

$$\tfrac{3}{5}x^2 - 6x = -4,$$

multiplying by the divisor 5,

$$3x^2 - 30x = -20,$$

dividing by the multiplier 3,

$$x^2 - 10x = -\tfrac{20}{3}.$$

If we now compare this equation with the general formula

$$x^2 + px = q,$$

we shall have

$$p = -10, \qquad q = -\tfrac{20}{3}.$$

109. In order to arrive at the solution of equations thus prepared, we should keep in mind what has been already observed (34), namely, that the square of a quantity, composed of two terms, always contains the square of the first term, double the product of the first term multiplied by the second, and the square of the second; consequently the first member of the equation

$$x^2 + 2ax + a^2 = b,$$

in which a and b are known quantities, is a perfect square, arising from $x + a$, and may be expressed thus,

$$(x + a)(x + a) = b.$$

If we take the square root of the first member and indicate that of the second, we have

$$x + a = \pm\sqrt{b},$$

an equation, which, considered with respect to x, is only of the first degree; and from which we obtain, by transposition,

$$x = -a \pm \sqrt{b}.$$

An equation of the second degree may therefore be easily resolved, whenever it can be reduced to the form

$$x^2 + 2ax + a^2 = b,$$

that is, whenever its first member is a perfect square.

But the first member of the general equation

$$x^2 + px = q,$$

contains already two terms, which may be considered as form-

ing part of the square of a binomial; namely, x^2, which is the square of the first term x, and px, or double the first multiplied by the second, which second is consequently only half of p, or $\frac{1}{2}p$. To complete the square of the binomial $x + \frac{1}{2}p$, there must be also the square of the second term, $\frac{1}{2}p$; but this square may be formed, since p and $\frac{1}{2}p$ are known quantities, and it may be added to the first member, if, to preserve the equality of the two members, it be added at the same time to the second; and this last member will still be a known quantity.

As the square of $\frac{1}{2}p$ is $\frac{1}{4}p^2$, if we add it to the two members of the proposed equation,

$$x^2 + px = q,$$

we shall have

$$x^2 + px + \tfrac{1}{4}p^2 = q + \tfrac{1}{4}p^2.$$

The first member of this result is the square of $x + \frac{1}{2}p$; taking then the root of the two members, we have

$$x + \tfrac{1}{2}p = \pm\sqrt{q + \tfrac{1}{4}p^2}, \quad (106);$$

by transposition this becomes

$$x = -\tfrac{1}{2}p \pm \sqrt{q + \tfrac{1}{4}p^2},$$

or which is the same thing

$$x = -\tfrac{1}{2}p + \sqrt{q + \tfrac{1}{4}p^2}.$$

and

$$x = -\tfrac{1}{2}p - \sqrt{q + \tfrac{1}{4}p^2}.$$

I have prefixed the sign $+$ to the second term $\frac{1}{2}p$, of the root of the first member of the above equation, because the second term of this member is positive; the sign $-$ is to be prefixed in the contrary case, because the square $x^2 - 2ax + a^2$ answers to the binomial $x - a$.

Any equation whatever of the second degree may be resolved, by referring it to the general formula,

$$x^2 + px = q;$$

or more expeditiously, by performing immediately upon the equation the operations represented under this formula, which, expressed in general terms, are as follows;

To make the first member of the proposed equation a perfect square, by adding to it, and also to the second, the square of half the given quantity, by which the first power of the unknown quantity is multiplied; then to extract the square root of each member, observing, that the root of the first member is composed of the unknown

quantity, and half of the given number, by which the unknown quantity in the second term is multiplied, taken with the sign of this quantity, and that the root of the second member must have the double sign $\pm$, *and be indicated by the sign* $\sqrt{\ }$, *if it cannot be obtained directly.*

See this illustrated by examples.

110. *To find a number such, that if it be multiplied by* 7, *and this product be added to its square, the sum will be* 44.

The number sought being represented by x, the equation will evidently be

$$x^2 + 7x = 44.$$ 1st form

In order to resolve this equation, we take $\frac{7}{2}$, half of the coefficient 7, by which x is multiplied; raising it to its square we obtain $\frac{49}{4}$; this added to each member gives

$$x^2 + 7x + \tfrac{49}{4} = 44 + \tfrac{49}{4};$$

reducing the second member to a single term, we have

$$x^2 + 7x + \tfrac{49}{4} = \tfrac{225}{4}.$$

The root of the first member, according to the rule given above, is $x + \frac{7}{2}$, and we find for that of the second $\frac{15}{2}$; whence arises the equation

$$x + \tfrac{7}{2} = \pm \tfrac{15}{2},$$

from which we obtain

$$x = -\tfrac{7}{2} \pm \tfrac{15}{2},$$

or

$$x = -\tfrac{7}{2} + \tfrac{15}{2} = \tfrac{8}{2} = 4,$$

$$x = -\tfrac{7}{2} - \tfrac{15}{2} = -\tfrac{22}{2} = -11.$$

The first value of x solves the question in the sense, in which it was enunciated, since we have by this value

$$\begin{array}{lr} & x^2 = 16 \\ & 7x = 28 \\ \hline \text{sum} & 44. \end{array}$$

As to the second value of x, since it is affected with the sign $-$, the term $7x$, which becomes

$$7 \times -11 = -77,$$

must be subtracted from x^2, so that the enunciation of the question resolved by the number 11 is this,

To find a number such, that 7 *times this number being subtracted from its square, the remainder will be* 44.

The negative value then here modifies the question in a manner, analogous to what takes place, as we have already seen, in equations of the first degree.

If we put the question, as enunciated above, into an equation, we obtain

$$x^2 - 7x = 44,$$

2 new form

this becomes, when resolved,

$$x^2 - 7x + \tfrac{49}{4} = 44 + \tfrac{49}{4},$$

$$x^2 - 7x + \tfrac{49}{4} = \tfrac{225}{4},$$

$$x - \tfrac{7}{2} = \pm \tfrac{15}{2},$$

$$x = \tfrac{7}{2} \pm \tfrac{15}{2},$$

$$x = \tfrac{22}{2} = 11,$$

$$x = \tfrac{7}{2} - \tfrac{15}{2} = -\tfrac{8}{2} = -4.$$

The negative value of x becomes positive, as it satisfies precisely the new enunciation, and the positive value, which does not thus satisfy it, becomes negative.

Hence we see, that in equations of the second degree, algebra unites under the same formula two questions, which have a certain analogy to each other.

111. Sometimes enunciations, which produce equations of the second degree, admit of two solutions. The following is an example;

To find a number such, that if 15 *be added to its square, the sum will be equal to* 8 *times this number.*

Let x be the number sought; the equation arising from the problem is then

$$x^2 + 15 = 8x.$$

This equation reduced to the form prescribed in art. 108, becomes

form

$$x^2 - 8x = -15,$$

$$x^2 - 8x + 16 = -15 + 16,$$

$$x^2 - 8x + 16 = 1,$$

$$x - 4 = \pm 1,$$

$$x = 4 \pm 1,$$

or

$$x = 5,$$

$$x = 3.$$

There are therefore two different numbers 5 and 3, which fulfil the conditions of the question.

112. Questions sometimes occur, which cannot be resolved precisely in the sense of the enunciation, and which require to be modified. This is the case, when the two roots of the equation are negative, as in the following example,

$$x^2 + 5x + 6 = 2.$$

This equation, which denotes, that *the square of the number sought, augmented by* 5 *times this number, and also by* 6, *will give a sum equal to* 2, evidently cannot be verified by addition, as is implied, since 6 already exceeds 2. Indeed, if we resolve it, we find successively

To form

$$x^2 + 5x = -4,$$
$$x^2 + 5x + \tfrac{25}{4} = \tfrac{25}{4} - 4 = \tfrac{9}{4},$$
$$x + \tfrac{5}{2} = \pm \tfrac{3}{2},$$
$$x = -\tfrac{5}{2} + \tfrac{3}{2} = -1,$$
$$x = -\tfrac{5}{2} - \tfrac{3}{2} = -4.$$

From the sign $-$, with which the numbers 1 and 4 are affected, it may be seen, that the term $5x$ must be subtracted from the others, and that the true enunciation for both values is,

To find a number such, that if 5 *times this number be subtracted from its square, and* 6 *be added to the remainder, the result will be* 2.

This enunciation furnishes the equation,

$$x^2 - 5x + 6 = 2,$$

which gives for x the two positive values 1 and 4.

113. Again, let the following problem be proposed;

To divide a number p *into two parts, the product of which shall be equal to* q.

If we designate one of these parts by x, the other will be expressed by $p - x$, and their product will be $px - x^2$; we have then the equation

$$px - x^2 = q,$$

or, changing the signs,

the form

$$x^2 - px = -q;$$

resolving this last, we find

$$x = \tfrac{1}{2}p \pm \sqrt{\tfrac{1}{4}p^2 - q}.$$

If now we suppose

$$p = 10, \qquad q = 21,$$

we have

$$x = 5 \pm \sqrt{25 - 21},$$

or $$x = 5 \pm 2,$$
$$x = 7,$$
$$x = 3,$$

that is, one of the parts will be 7, and the other consequently $10 - 7$, or 3.

If on the contrary, we take 3 for x, the other part will be

10 — 3 or 7; so that the enunciation, as it stands, admits, strictly speaking, of only one solution, since the second amounts simply to a change in the order of the parts.

If we examine carefully the value of x in the question we have been considering, we shall see that we cannot take any numbers indifferently for p and q, for if q exceed $\frac{p^2}{4}$, or the square of $\frac{1}{2}p$, the quantity $\frac{p^2}{4} - q$ becomes negative, and we are presented with that species of absurdity mentioned in art. 107.

If we take, for example,

$$p = 10 \text{ and } q = 30,$$

we have

$$x = 5 \pm \sqrt{25 - 30} = 5 \pm \sqrt{-5};$$

the problem then with these assumptions is impossible.

114. The absurdity of questions, which lead to imaginary roots, is discovered only by the result, and we may wish to determine by characters, which are found nearer to the enunciation, in what consists the absurdity of the problem, which gives rise to that of the solution; this we shall be enabled to do by the following consideration.

Let d be the difference of the two parts of the proposed number; the greater part will be $\frac{p}{2} + \frac{d}{2}$, the less $\frac{p}{2} - \frac{d}{2}$ (3); but it has been proved (29, 30, & 34) that

$$\left(\frac{p}{2} + \frac{d}{2}\right)\left(\frac{p}{2} - \frac{d}{2}\right) = \frac{p^2}{4} - \frac{d^2}{4};$$

therefore, the product of the two parts of the proposed number, whatever they may be, will always be less than $\frac{p^2}{4}$, or than the square of half their sum, so long as d is any thing but zero; when d is nothing, each of the two parts being equal to $\frac{p}{2}$, their product will be only $\frac{p^2}{4}$. It is then absurd to require it to be greater; and it is just, that algebra should answer in a manner contradictory to established principles, and thereby show, that what is sought does not exist.

What has been proved concerning the equation

$$x^2 - px = -q,$$

furnished by the preceding question, is true of all those of the

second degree, where q is negative in the second member, the only equations, which produce imaginary roots, since the term $\frac{p^2}{4}$, placed under the radical sign, preserves always the sign $+$, whatever may be that of p. Indeed, it is evident that the equation

$$x^2 + px = -q, \text{ or } x^2 + px + q = 0,$$

will admit of no positive solution, since the first member contains only affirmative terms; and, to ascertain whether the unknown quantity x can be negative, we have only to change x into $-y$. The unknown quantity y would then have positive values, which would be furnished by the equation

$$y^2 - py + q = 0, \quad \text{or} \quad y^2 - py = -q,$$

which is precisely the same as that in the preceding article; but as the values of x can be real, only when those of y would be so, they become therefore imaginary in the case under consideration, when q exceeds $\frac{p^2}{4}$.

It will be perceived then from what has been said, how, and for what reason, *when the known term of an equation of the second degree is negative in the second member, and greater than the square of half the coefficient of the first power of the unknown quantity, this equation can have only imaginary roots.*

115. The expressions

$$\sqrt{-b}, \quad a + \sqrt{-b},$$

and, in general, those, which involve the square root of a negative quantity, are called *imaginary quantities.** They are mere symbols of absurdity, that take the place of the value, which we should have obtained, if the question had been possible.

They are not, however, to be neglected in the calculation, because it sometimes happens, that when they are combined according to certain laws, the absurdity disappears, and the result becomes real. Examples of this kind will be found in the *Supplement* to this treatise.

116. As it is important, that learners should have just ideas respecting all those analytical *facts*, which appear to be derived from familiar notions, I have thought it proper to add some observations to what has been said (106), on the necessity of admitting two solutions in equations of the second degree.

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* It would be more correct to say, *imaginary expressions,* or *symbols,* as they are not quantities.

I shall show that, *if there exists a quantity* a, *which, substituted in the place of* x, *verifies the equation of the second degree*, $x^2 + px = q$, *and is consequently the value of* x, *this unknown quantity will still have another value.* Now, if we substitute a for x, the result will be $a^2 + pa = q$; and since, by supposition, a represents the value of x, q will be necessarily equal to the quantity $a^2 + pa$; we may then write this quantity in the place of q, in the proposed equation, which thus becomes

$$x^2 + px = a^2 + pa.$$

Transposing all the terms of the second member, we have

$$x^2 + px - a^2 - pa = 0,$$

which may be written,

$$x^2 - a^2 + p(x - a) = 0;$$

and because

$$x^2 - a^2 = (x + a)(x - a), \quad (34),$$

it is obvious, at once, that the first member is divisible by $x - a$, and will give an exact quotient, namely, $x + a + p$; we have then,

$$x^2 + px - q = x^2 - a^2 + p(x - a) = (x - a)(x + a + p).$$

Now it is evident, that a product is equal to zero, when any one of its factors whatever becomes nothing; we shall have then

$$(x - a)(x + a + p) = 0,$$

not only when $x - a = 0$, which gives

$$x = a,$$

but also when $x + a + p = 0$, from which is deduced

$$x = -a - p.$$

Therefore, if a is one of the values of x, $-a-p$ will necessarily be the other.

This result agrees with the two values comprehended in the formula

$$x = -\tfrac{1}{2}p \pm \sqrt{q + \tfrac{1}{4}p^2};$$

for if we take for a the first value, $-\frac{1}{2}p + \sqrt{q + \frac{1}{4}p^2}$, we obtain for the other

$$-a - p = +\tfrac{1}{2}p - \sqrt{q + \tfrac{1}{4}p^2} - p = -\tfrac{1}{2}p - \sqrt{q + \tfrac{1}{4}p^2},$$

which is in fact the second value.

These remarks contain the germ of the general theory of equations of whatever degree, as will appear hereafter, when the subject will be resumed.

117. The difficulty of putting a problem into an equation, is the same in questions involving the second and higher powers, as

in those involving only the first, and consists always in disentangling and expressing distinctly in algebraic characters all the conditions comprehended in the enunciation. The preceding questions present no difficulty of this sort; and, although the learner is supposed to be well exercised in those of the first degree, I shall proceed to resolve a few questions, which will furnish occasion for some instructive remarks.

A person employed two labourers, allowing them different wages; the first received, at the end of a certain number of days, 96 francs, and the second, having worked six days less, received only 54 francs; if this last had worked the whole number of days, and the other had lost six days, they would both have received the same sum; it is required to find how many days each worked, and what sum each received for a day's work.

This problem, which at first view appears to contain several unknown quantities, may be easily solved by means of one, because the others may be readily expressed by this.

If x represent the number of days' work of the first labourer, $x - 6$ will be the number of days' work of the second,

$\frac{96^{\text{fr.}}}{x}$ will be the daily wages of the first,

$\frac{54}{x-6}$ the daily wages of the second;

if this last had worked x days, he would have earned

$$x \times \frac{54}{x-6} \text{ or } \frac{54x}{x-6},$$

and the first, working $x - 6$ days, would have received only

$$(x-6)\frac{96}{x}, \text{ or } \frac{96(x-6)}{x}.$$

The equation of the problem then will be

$$\frac{54x}{x-6} = \frac{96(x-6)}{x}.$$

The first step is to make the denominators disappear; the equation then becomes

$$54x^2 = 96(x-6)(x-6).$$

As the numbers 54 and 96 are both divisible by 6, the result may be simplified by division, we shall then have

$$9x^2 = 16(x-6)(x-6).$$

This last equation may be prepared for solution according to the rule given art. 108, but as the object of this rule is to enable us

with more facility to extract the root of each member of the equation proposed, it is here unnecessary, because the two members are already presented under the form of squares; for it is evident, that $9x^2$ is the square of $3x$, and $16(x-6)(x-6)$ the square of $4(x-6)$. We have then

$$3x = \pm 4(x-6);$$

from which may be deduced

$$3x = 4x - 24,\quad x = 24,$$
$$3x = -4x + 24,\quad x = \tfrac{24}{7}.$$

By the first solution, the first labourer worked 24 days, and consequently earned $\frac{96}{24}$ or 4 francs per day, while the second worked only 18 days, and received $\frac{54}{18}$ or 3 francs per day.

The second solution answers to another numerical question, connected with the equation under consideration, in a manner analogous to what was noticed in art. 111.

118. *A banker receives two notes against the same person; the first of 550 francs, payable in seven months, the second of 720 francs, payable in four months, and gives for both the sum of 1200 francs; it is required to find, what is the annual rate of interest, according to which these notes are discounted.*

In order to avoid fractions in expressing the interest for seven months and four months, we shall represent by $12x$ the interest of 100 francs for one year; the interest for one month will then be x. The present value of the first note will accordingly be found by the proportion,

$$100 + 7x : 100 :: 550 : \frac{55000}{100 + 7x} \quad (\textit{Arith. } 120);$$

and the present value of the second note by the proportion,

$$100 + 4x : 100 :: 720 : \frac{72000}{100 + 4x}.$$

By uniting these values, we obtain for the equation of the problem,

$$\frac{55000}{100 + 7x} + \frac{72000}{100 + 4x} = 1200.$$

Dividing each of the members by 200, we have

$$\frac{275}{100 + 7x} + \frac{360}{100 + 4x} = 6;$$

making the denominators disappear, we find successively,

$$275(100 + 4x) + 360(100 + 7x) = 6(100 + 7x)(100 + 4x),$$
$$27500 + 1100x + 36000 + 2520x = 60000 + 6600x + 168x^2,$$

which may be reduced to

$$168x^2 + 2980x = 3500;$$

dividing by 2, we obtain

$$84x^2 + 1490x = 1750,$$

which gives

$$x^2 + \frac{1490}{84}x = \frac{1750}{84}.$$

Comparing this equation with the formula,

$$x^2 + px = q,$$

we have

$$p = \frac{1490}{84}, \qquad q = \frac{1750}{84};$$

and the expression

$$x = -\frac{1}{2}p \pm \sqrt{\frac{p^2}{4} + q},$$

becomes

$$x = -\frac{745}{84} \pm \sqrt{\frac{745.745}{84.84} + \frac{1750}{84}}.$$

Reducing the fractions, we have

$$\frac{745.745 + 1750.84}{84.84} = \frac{702025}{84.84};$$

then, since the denominator of this fraction is a perfect square, we have only to extract the square root of its numerator. If we stop at thousandths, we find 837,869, for the root of 702025; this, taken with the denominator 84, gives for the values of x

$$x = -\frac{745}{84} + \frac{837,869}{84} = \frac{92,869}{84},$$

$$x = -\frac{745}{84} - \frac{837,869}{84} = -\frac{1582,869}{84}.$$

The first of these values is the only one, which solves the question in the sense, in which it was enunciated. Dividing the denominator of this fraction by 12, we have (*Arith.* 54.)

$$12x = \frac{92,869}{7} = 13,267;$$

that is, the annual interest is at the rate of 13,27 nearly.

119. The following question deserves attention on account of the character, which the expression for the unknown quantity presents.

To divide a number into two parts, the squares of which shall be in a given ratio.

Let a be the given number,

m the ratio of the squares of its two parts,

x one of these parts;

the other will be $a - x$.

We shall then have, according to the enunciation,

$$\frac{x^2}{(a-x)(a-x)} = m.$$

This may be resolved in two ways; we may either reduce it to the form $x^2 + px = q$, and then resolve it by the common method; or since the fraction

$$\frac{x^2}{(a-x)(a-x)}$$

is a square, the numerator and denominator being each a square, we thence conclude at once,

$$\frac{x}{a-x} = \pm \sqrt{m},$$

$$x = \pm (a-x)\sqrt{m}.$$

By resolving separately the two equations of the first degree comprehended in this formula, namely,

$$x = +(a-x)\sqrt{m},$$

$$x = -(a-x)\sqrt{m},$$

we have

$$x = \frac{a\sqrt{m}}{1+\sqrt{m}},$$

$$x = \frac{-a\sqrt{m}}{1-\sqrt{m}}.$$

By the first solution, the second part of the number proposed is

$$a - \frac{a\sqrt{m}}{1+\sqrt{m}} = \frac{a + a\sqrt{m} - a\sqrt{m}}{1+\sqrt{m}} = \frac{a}{1+\sqrt{m}};$$

and the two parts,

$$\frac{a\sqrt{m}}{1+\sqrt{m}} \text{ and } \frac{a}{1+\sqrt{m}},$$

are both, as the enunciation requires, less than the number proposed.

By the second solution we have

$$a + \frac{a\sqrt{m}}{1-\sqrt{m}} = \frac{a - a\sqrt{m} + a\sqrt{m}}{1-\sqrt{m}} = \frac{a}{1-\sqrt{m}};$$

and the two parts are

$$-\frac{a\sqrt{m}}{1-\sqrt{m}} \text{ and } \frac{a}{1-\sqrt{m}}.$$

Their signs being opposite, the number a is strictly no longer their sum, but their difference.

If we make $m = 1$, that is, if we suppose that the squares of the two parts sought are equal, we have

$$\sqrt{m} = 1;$$

and the first solution will give two equal parts,

$$\frac{a}{2}, \quad \frac{a}{2},$$

a conclusion, that is self-evident, while the second solution gives for the results two infinite quantities (68), namely,

$$\frac{-a}{1-1} \text{ or } \frac{-a}{0}, \text{ and } \frac{a}{1-1} \text{ or } \frac{a}{0}.$$

This is necessary, for it is only by considering two quantities infinitely great, with respect to their difference a, that we can suppose the ratio of their squares equal to unity.

Now, let there be the two quantities x, and $x - a$, the ratio of their squares will be

$$\frac{x^2}{x^2 - 2ax + a^2};$$

dividing the two terms of this fraction by x^2, we obtain

$$\frac{1}{1 - \frac{2a}{x} + \frac{a^2}{x^2}};$$

but it is evident, that the greater the number x, the less will be the fractions $\frac{2a}{x}$, $\frac{a^2}{x^2}$, and the more nearly will the above ratio approach to $\frac{1}{1}$, or 1.

120. Now in order to compare the general method with that, which we have just employed, we develope the equation

$$\frac{x^2}{(a-x)(a-x)} = m;$$

and we have, successively,

$$x^2 = m(a-x)(a-x),$$
$$x^2 = a^2 m - 2amx + mx^2,$$
$$x^2 - mx^2 + 2amx = a^2 m,$$
$$(1-m)x^2 + 2amx = a^2 m,$$
$$x^2 + \frac{2amx}{1-m} = \frac{a^2 m}{1-m},$$

making $\qquad p = \frac{2am}{1-m}, \qquad q = \frac{a^2 m}{1-m},$

the general formula gives

$$x = -\frac{am}{1-m} \pm \sqrt{\frac{a^2 m^2}{(1-m)(1-m)} + \frac{a^2 m}{1-m}}.$$

These values of x appear very different from those, which were found above; yet they may be reduced to the same; and in this consists the utility of the example, on which I am employed. It will serve to show the importance of those transformations, which different algebraic operations produce in the expression of quantities.

We must first reduce the two fractions comprehended under the radical sign to a common denominator. This may be done by multiplying the two terms of the second by $1 - m$, we have then

$$\frac{a^2 m^2}{(1-m)(1-m)} + \frac{a^2 m}{1-m} = \frac{a^2 m^2 + a^2 m\,(1-m)}{(1-m)\,(1-m)} =$$

$$\frac{a^2 m^2 + a^2 m - a^2 m^2}{(1-m)\,(1-m)} = \frac{a^2 m}{(1-m)\,(1-m)}.$$

The denominator being a square, it is only necessary to extract the root of the numerator, we then have

$$\sqrt{\frac{a^2 m^2}{(1-m)\,(1-m)} + \frac{a^2 m}{1-m}} = \frac{\sqrt{a^2 m}}{1-m};$$

but the expression $\sqrt{a^2 m}$ may be further simplified.

It is evident that the square of a product is composed of the product of the squares of each of its factors, for example,

$$bcd \times bcd = b^2 c^2 d^2,$$

and consequently the root of $b^2 c^2 d^2$ is simply the product of the roots b, c, and d, of the factors b^2, c^2, and d^2. Applying this principle to the product $a^2 m$, we see that its root is the product of a, the root of a^2, by $\sqrt{m}$, which denotes the root of m, or that

$$\sqrt{a^2 m} = a\sqrt{m}.$$

It follows from these different transformations, that

$$x = -\frac{am}{1-m} \pm \frac{a\sqrt{m}}{1-m},$$

or

$$x = -\frac{am - a\sqrt{m}}{1-m},$$

$$x = -\frac{am + a\sqrt{m}}{1-m}.$$

These expressions, however simple, are still not the same as those given in the preceding article; if, moreover, we seek to verify them for the case, in which $m = 1$, they become

$$x = \frac{-a+a}{1-1} = \frac{0}{0},$$

$$x = \frac{-a-a}{1-1} = \frac{-2a}{0}.$$

We find, in the second, the symbol of infinity, as in the preceding article, but the first presents this indeterminate form, $\frac{0}{0}$, of which we have already seen examples in articles 69 and 70; and before we pronounce upon its value, it is proper to examine, whether it does not belong to the case stated in art. 70; whether there is not some factor common to the numerator and denominator, which the supposition of $m = 1$ renders equal to zero.

The expression $\dfrac{-am + a\sqrt{m}}{1-m}$

may be resolved into

$$\frac{a(-m+\sqrt{m})}{1-m} = \frac{a(\sqrt{m}-m)}{1-m}.$$

It is here evident, that the numerator does not become 0, except by means of the factor $\sqrt{m} - m$; we must therefore examine, whether this last has not some factor in common with the denominator $1 - m$. In order to avoid the inconvenience, arising from the use of the radical sign, let us make $\sqrt{m} = n$, then taking the squares, we have $m = n^2$; the quantities, therefore,

$$\sqrt{m} - m \text{ and } 1 - m$$

become $\qquad n - n^2$ and $1 - n^2$,

but $n - n^2 = n(1-n)$, and $1 - n^2 = (1-n)(1+n)$ (34); restoring to the place of n its value $\sqrt{m}$, we have

$$\sqrt{m} - m = (1-\sqrt{m})\sqrt{m},$$

$$1 - m = (1-\sqrt{m})(1+\sqrt{m}),$$

and consequently,

$$\frac{a(\sqrt{m}-m)}{1-m} = \frac{a(1-\sqrt{m})\sqrt{m}}{(1-\sqrt{m})(1+\sqrt{m})} = \frac{a\sqrt{m}}{1+\sqrt{m}},$$

a result the same, as that found in art. 119.*

In the same manner we may reduce the second value of x, observing that

$$\frac{-a\sqrt{m}-am}{1-m} = \frac{-a(1+\sqrt{m})\sqrt{m}}{(1-\sqrt{m})(1+\sqrt{m})} = \frac{-a\sqrt{m}}{1-\sqrt{m}},$$

as in art. 119.*

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* The example, which I have given at some length, corresponds with a problem resolved by Clairaut, in his Algebra, the enunciation

It will be seen without difficulty, that we might have avoided radical expressions in the preceding calculations, by taking m^2 to represent the ratio, which the squares of the two parts of the proposed number have to each other; m would then have been the square root, which may always be considered as known, when the square is known; but we could not have perceived from the beginning the object of such a change in a given term, of which algebraists often avail themselves, in order to render calculations more simple. It is recommended to the learner, therefore, to go over the solution again, putting m^2 in the place of m.

Of the Extraction of the Square Root of Algebraic Quantities.

121. We have sufficiently illustrated, by the preceding example, the manner of conducting the solution of literal questions. We have given also an instance of a transformation, namely, that of $\sqrt{a^2 m}$ into $a\sqrt{m}$, which is worthy of particular attention; since, by means of it, we have been able to reduce the factors, contained under a radical sign, to the smallest number possible, and thus to simplify very much the extraction of the remaining part of the root.

This transformation consists in *taking the roots of all the factors which are squares, and writing them without the radical sign, as multipliers of the radical quantity, and retaining under the radical sign all those factors, which are not squares.*

This rule supposes, that the student is already able to determine, whether an algebraic quantity is a square, and is acquainted with the method of extracting the root of such a quantity. In order to this, it is necessary to distinguish simple quantities from polynomials.

122. It is evident, from the rule given for the exponents in

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of which is as follows; *To find on the line, which joins any two luminous bodies, the point where these two bodies shine with equal light.* I have divested this problem of the physical circumstances, which are foreign to the object of this work, and which only divert the attention from the character of the algebraic expressions. These expressions are very remarkable in themselves, and for this reason I have developed them more fully, than they were done in the work referred to.

multiplication, that the *second power of any quantity has an exponent double that of this quantity.*

We have, for example,

$$a^1 \times a^1 = a^2, \quad a^2 \times a^2 = a^4, \quad a^3 \times a^3 = a^6, \text{ \&c.}$$

It follows then, that *every factor, which is a square, must have an exponent which is an even quantity, and that the root of this factor is found by writing its letter with an exponent equal to half the original exponent.*

Thus we have

$$\sqrt{a^2} = a^1 \text{ or } a, \quad \sqrt{a^4} = a^2, \quad \sqrt{a^6} = a^3, \text{ \&c.}$$

With respect to numerical factors, their roots are extracted, when they admit of any, by the rules already given.

Whence the factors a^6, b^4, c^2, in the expression

$$\sqrt{64\, a^6\, b^4\, c^2},$$

are squares, and the number 64 is the square of 8; therefore, *as the expression proposed is the product of factors, which are squares, it will have for a root the product of the roots of these several factors* (121); and, consequently,

$$\sqrt{64\, a^6\, b^4\, c^2} = 8\, a^3\, b^2\, c.$$

123. In other cases, different from the above, *we must endeavour to resolve the proposed quantity, considered as a product, into two other products, one of which shall contain only such factors as are squares, and the other those factors which are not squares.* To effect this, we must consider each of the quantities separately.

Let there be, for example,

$$\sqrt{72\, a^4\, b^3\, c^5}.$$

We see that among the divisors of 72, the following are perfect squares, namely, 4, 9, and 36; if we take the greatest, we have

$$72 = 36 \times 2.$$

As the factor a^4 is a square, we separate it from the others; passing then to the factor b^3, which is not a square, since 3 is an odd number, we observe that this factor may be resolved into two others, b^2 and b, the first of which is a square; we have then

$$b^3 = b^2 \,.\, b;$$

it is obvious also that

$$c^5 = c^4 \,.\, c.$$

By proceeding in the same manner with every letter, whose exponent is an odd number, the quantity is resolved thus,

$$72\, a^4\, b^3\, c^5 = 36 \,.\, 2\, a^4\, b^2 \,.\, b\, c^4 \,.\, c;$$

by collecting the factors, which are squares, it becomes

$$36\,a^4\,b^2\,c^4 \times 2\,b\,c.$$

Lastly, taking the root of the first product and indicating that of the second, we have

$$\sqrt{72\,a^4\,b^3\,c^5} = 6\,a^2\,b\,c^2\,\sqrt{2\,b\,c}.$$

See some examples of this kind of reduction, with the steps, by which they are performed;

$$\sqrt{\frac{a^3}{b}} = \sqrt{a^2\,\frac{a}{b}} = a\sqrt{\frac{a}{b}} = a\sqrt{\frac{a\,b}{b^2}} = \frac{a}{b}\sqrt{a\,b};$$

$$6\sqrt{\frac{75}{98}\,a\,b^2} = 6\sqrt{\frac{25\,.\,3\,a\,b^2}{49\,.\,2}} = 6\sqrt{\frac{25\,b^2\,.\,3\,a}{49\,.\,2}} =$$

$$\frac{6\,.\,5}{7}\,b\sqrt{\frac{3\,a}{2}} = \frac{30\,b}{7}\sqrt{\frac{3\,a}{2}};$$

$$\sqrt{\frac{a^2\,m^2}{n^2} + \frac{a^2\,m}{n}} = \sqrt{\frac{a^2\,m^2 + a^2\,m\,n}{n^2}} =$$

$$\sqrt{\frac{a^2}{n^2}\,(m^2 + m\,n)} = \frac{a}{n}\sqrt{m^2 + m\,n}.$$

It will be seen by the first of these examples, that the denominator of an algebraic fraction may be taken from under the radical sign by being made a complete square, in the same manner as we reduce the root of a numerical fraction (104.)

124. We now proceed to the extraction of the square root of polynomials. It must here be recollected, that no binomial is a perfect square, because every simple quantity raised to a square produces only a simple quantity, and the square of a binomial always contains three parts (34).

It would be a great mistake to suppose the binomial $a + b$ to be the square root of $a^2 + b^2$, although taken separately, a is the root of a^2, and b that of b^2; for the square of $a + b$, or $a^2 + 2\,a\,b + b^2$, contains the term $+ 2\,a\,b$, which is not found in the expression $a^2 + b^2$.

Let there be the trinomial

$$24\,a^2\,b^3\,c + 16\,a^4\,c^2 + 9\,b^6.$$

In order to obtain from this expression the three parts, which compose the square of a binomial, we must arrange it with reference to one of its letters, the letter a, for example; it then becomes $\quad 16\,a^4\,c^2 + 24\,a^2\,b^3\,c + 9\,b^6.$

Now, whatever be the square root sought, if we suppose it arranged with reference to the same letter a, the square of its first term must necessarily form the first term, $16\,a^4\,c^2$, of the proposed quantity; double the product of the first term of the root by the second must give the second term, $24\,a^2\,b^3\,c$, of the proposed quantity; and the square of the last term of the root must give exactly the last term, $9\,b^6$, of the proposed quantity. The operation may be exhibited, as follows;

$$\begin{array}{rrr|l}
16\,a^4\,c^2 & +\,24\,a^2\,b^3\,c & +\,9\,b^6 & 4\,a^2\,c + 3\,b^3 \text{ root} \\
-\,16\,a^4\,c^2 & & & 8\,a^2\,c + 3\,b^3 \\
\hline
 & +\,24\,a^2\,b^3\,c & +\,9\,b^6 & \\
 & -\,24\,a^2\,b^3\,c & -\,9\,b^6 & \\
\hline
 & 0 & 0 &
\end{array}$$

We begin by finding the square root of the first term, $16\,a^4\,c^2$, and the result $4\,a^2\,c$ (122) is the first term of the root, which is to be written on the right, upon the same line with the quantity, whose root is to be extracted.

We subtract from the proposed quantity, the square, $16\,a^4\,c^2$, of the first term, $4\,a^2\,c$, of the root; there remain then only the two terms $24\,a^2\,b^3\,c + 9\,b^6$.

As the term $24\,a^2\,b^3\,c$ is double the product of the first term of the root, $4\,a^2\,c$, by the second, we obtain this last, by dividing $24\,a^2\,b^3\,c$ by $8\,a^2\,c$, double of $4\,a^2\,c$, which is written below the root; the quotient $3\,b^3$ is the second term of the root.

The root is now determined; and, if it be exact, the square of the second term will be $9\,b^6$, or rather, double of the first term of the root $8\,a^2\,c$ together with the second $3\,b^3$, multiplied by the second, will reproduce the two last terms of the square (91); therefore we write $+\,3\,b^3$ by the side of $8\,a^2\,c$, and multiply $8\,a^2\,c + 3\,b^3$ by $3\,b^3$; after the product is subtracted from the two last terms of the quantity proposed, nothing remains; and we conclude, that this quantity is the square of $4\,a^2\,c + 3\,b^3$.

It is evident that the same reasoning and the same process may be applied to all quantities composed of three terms.

125. When the quantity, whose root is to be extracted, has more than three terms, it is no longer the square of a binomial; but if we suppose it the square of a trinomial, $m + n + p$, and represent by l the sum $m + n$, this trinomial becoming now $l + p$, its square will be

$$l^2 + 2\,l\,p + p^2,$$

in which the square l^2 of the binomial $m + n$, produces, when developed, the terms $m^2 + 2mn + n^2$. Now, after we have arranged the proposed quantity, the first term will evidently be the square of the first term of the root, and the second will contain double the product of the first term of the root by the second of this root; we shall then obtain this last by dividing the second term of the proposed quantity by double the root of the first. Knowing then the two first terms of the root sought, we complete the square of these two terms, represented here by l^2; subtracting this square from the proposed quantity, we have for a remainder

$$2lp + p^2,$$

a quantity, which contains double the product of l, or of the first binomial $m + n$, by the remainder of the root, plus the square of this remainder. It is evident, therefore, that we must proceed with this binomial as we have done with the first term m of the root.

Let there be, for example, the quantity
$64a^2bc + 25a^2b^2 - 40a^3b + 16a^4 + 64b^2c^2 - 80ab^2c$;
we arrange it with reference to the letter a, and make the same disposition of the several parts of the operation as in the above example.

$$\begin{array}{ll|l}
 & 16a^4 - 40a^3b + 25a^2b^2 - 80ab^2c + 64b^2c^2 & 4a^2 - 5ab + 8bc \\
 & \qquad\qquad + 64a^2bc & \overline{8a^2 - 5ab} \\
 & -16a^4 & 8a^2 - 10ab + 8bc \\
\hline
\text{1st rem.} & -40a^3b + 25a^2b^2 - 80ab^2c + 64b^2c^2 & \\
 & \qquad + 64a^2bc & \\
 & +40a^3b - 25a^2b^2 & \\
\hline
\text{2d rem.} \ldots\ldots & +64a^2bc - 80ab^2c + 64b^2c^2 & \\
 & -64a^2bc + 80ab^2c - 64b^2c^2 & \\
\hline
 & \qquad 0 \qquad\quad 0 \qquad\quad 0 &
\end{array}$$

We extract the square root of the first term $16a^4$, and obtain $4a^2$ for the first term of the root sought, the square of which is to be subtracted from the proposed quantity.

We double the first term of the root, and write the result, $8a^2$, under the root; dividing by this the term $-40a^3b$, which begins the first remainder, we have $-5ab$ for the second term of the root; this is to be placed by the side of $8a^2$, we then multiply the whole by this second term, and subtract the result from the remainder, upon which we are employed.

Thus we have subtracted from the proposed quantity the square of the binomial $4a^2 - 5ab$; the second remainder can contain only double the product of this binomial, by the third term of the root, together with the square of this term; we take then double the quantity $4a^2 - 5ab$, or

$$8a^2 - 10ab,$$

which is written under $8a^2 - 5ab$, and constitutes the divisor to be used with the second remainder; the first term of the quotient, which is $8bc$, is the third of the root.

This term we write by the side of $8a^2 - 10ab$, and multiply the whole expression by it; the product being subtracted from the remainder under consideration, nothing is left; the quantity proposed, therefore, is the square of

$$4a^2 - 5ab + 8bc.$$

The above operation, which is perfectly analogous to that, which has been already applied to numbers, may be extended to any length we please.

Of the formation of Powers and the extraction of their Roots.

126. The arithmetical operation, upon which the resolution of equations of the second degree depends, and by which we ascend from the square of a quantity to the quantity, from which it is derived, or to the square root, is only a particular case of a more general problem, namely, *to find a number, any power of which is known.* The investigation of this problem leads to a result, that is still termed a root, the different kinds being called degrees, but the process is to be understood only by a careful examination of the steps by which a power is obtained, one operation being the reverse of the other, as we observe with respect to division and multiplication, with which it will soon be perceived that this subject has other relations.

It is by multiplication, that we arrive at the powers of entire numbers (24), and it is evident, that those of fractions also are formed by raising the numerator and denominator to the power proposed (96).

So also the root of a fraction, of whatever degree, is obtained by taking the corresponding root of the numerator and that of the denominator.

As algebraic symbols are of great use in expressing every thing, which relates to the composition and decomposition of

quantities, I shall first consider how the powers of algebraic expressions are formed, those of numbers being easily found by the methods that have already been given (24).

Table of the first Seven Powers of Numbers from 1 *to* 9.

1st	1	2	3	4	5	6	7	8	9
2d	1	4	9	16	25	36	49	64	81
3d	1	8	27	64	125	216	343	512	729
4th	1	16	81	256	625	1296	2401	4096	6561
5th	1	32	243	1024	3125	7776	16807	32768	59049
6th	1	64	729	4096	15625	46656	117649	262144	531441
7th	1	128	2187	16384	78125	279936	823543	2097152	4782969

This table is intended particularly to show with what rapidity the higher powers of numbers increase, a circumstance that will be found to be of great importance hereafter; we see, for instance, that the seventh power of 2 is 128, and that of 9 amounts to 4782969.

It will hence be readily perceived that the powers of fractions, properly so called, decrease very rapidly, since the powers of the denominator become greater and greater in comparison with those of the numerator. The seventh power of $\frac{1}{2}$, for example, is $\frac{1}{128}$, and that of $\frac{1}{9}$ is only

$$\frac{1}{4782969}.$$

127. It is evident from what has been said, that in a product each letter has for an exponent the sum of the exponents of its several factors (26), that *the power of a simple quantity is obtained by multiplying the exponent of each factor by the exponent of this power.*

The third power of $a^2 b^3 c$, for example, is found by multiplying the exponents 2, 3, and 1, of the letters a, b, and c, by 3, the exponent of the power required; we have then $a^6 b^9 c^3$; the operation may be thus represented,

$$a^2 b^3 c \times a^2 b^3 c \times a^2 b^3 c = a^{2 \cdot 3} b^{3 \cdot 3} c^{1 \cdot 3}.$$

If the proposed quantity have a numerical coefficient, this coefficient must also be raised to the same power; thus the fourth power of $3\,a\,b^2\,c^5$, is

$$81\,a^4\,b^8\,c^{20},$$

128. With respect to the signs, with which simple quantities may be affected, it must be observed, that *every power, the exponent of which is an even number, has the sign* $+$, *and every power, the exponent of which is an odd number, has the same sign as the quantity from which it is formed.*

In fact, powers of an even degree arise from the multiplication of an even number of factors; and the signs $-$, combined two and two in the multiplication, always give the sign $+$ in the product (31). On the contrary, if the number of factors is uneven, the product will have the sign $-$, when the factors have this sign, since this product will arise from that of an even number of factors, multiplied by a negative factor.

129. In order to ascend from the power of a quantity, to the root from which it is derived, we have only to reverse the rules given above, that is, *to divide the exponent of each letter by that, which marks the degree of the root required.*

Thus we find the *cube root*, or *the root of the third degree*, of the expression $a^6 b^9 c^3$, by dividing the exponents 6, 9, and 3, by 3, which gives

$$a^2 b^3 c.$$

When the proposed expression has a numerical coefficient, its root must be taken for the coefficient of the literal quantity, obtained by the preceding rule.

If it were required, for example, to find the fourth root of $81\, a^4 b^8 c^{20}$, we see, by referring to table, art. 126, that 81 is the fourth power of 3; then, dividing the exponent of each of the letters by 4, we obtain for the result

$$3\, a\, b^2 c^5.$$

When the root of the numerical coefficient cannot be found by the table inserted above, it must be extracted by the methods to be given hereafter.

130. It is evident, that the roots of the literal part of simple quantities can be extracted, only when each of the exponents is divisible by that of the root; in the contrary case, we can only indicate the arithmetical operation, which is to be performed, whenever numbers are substituted in the place of the letters.

We use for this purpose the sign $\sqrt{\ }$; but to designate the degree of the root, we place the exponent as in the following expressions,

$$\sqrt[3]{a}, \quad \sqrt[5]{a^2},$$

the first of which represents the cube root, or the root of the third degree of a, and the second the fifth root of a^2.

We may often simplify radical expressions of any degree whatever, by observing, according to art. 127, that *any power of a product is made up of the product of the same power of each of the factors*, and that, consequently, *any root of a product is made up of the product of the roots of the same degree of the several factors.* It follows from this last principle, that, *if the quantity placed under the radical sign have factors, which are exact powers of the degree denoted by this sign, the roots of these factors may be taken separately, and their product multiplied by the root of the other factors indicated by the sign.*

Let there be, for example,

$$\sqrt[5]{96\,a^5\,b^7\,c^{11}}.$$

It may be seen that,

$$96 = 32 \times 3 = 2^5 \,.\, 3,$$

that $\quad a^5 \quad$ is the fifth power of a,

that $\quad b^7 = b^5 \,.\, b^2,$

that $\quad c^{11} = c^{10} \,.\, c;$

we have then

$$96\,a^5\,b^7\,c^{11} = 2^5\,a^5\,b^5\,c^{10} \times 3\,b^2\,c.$$

As the first factor, $2^5\,a^5\,b^5\,c^{10}$, has for its fifth root the quantity $2\,a\,b\,c^2$, the expression becomes

$$\sqrt[5]{96\,a^5\,b^7\,c^{11}} = 2\,a\,b\,c^2\,\sqrt[5]{3\,b^2\,c}.$$

131. As every even power has the sign $+$ (128), a quantity, affected with the sign $-$, cannot be a power of a degree denoted by an even number, and it can have no root of this degree. It follows from this, *that every radical expression of a degree which is denoted by an even number, and which involves a negative quantity, is imaginary*, thus

$$\sqrt[4]{-a},\ \sqrt[6]{-a^4},\ b + \sqrt[8]{-ab^7},$$

are imaginary expressions.

We cannot, therefore, either exactly or by approximation, assign for a degree, the exponent of which is an even number, any roots but those of positive quantities, and *these roots may be affected indifferently with the sign $+$ or $-$*, because, in either case, they will equally reproduce the proposed quantity with the sign $+$, and we do not know to which class they belong.

The same cannot be said of degrees expressed by an odd num-

ber, for here the powers have the same sign as their roots (128); and *we must give to the roots of these degrees the sign, with which the power is affected;* and no imaginary expressions occur.

132. It is proper to observe, that the application of the rule given in art. 129, for the extraction of the roots of simple quantities, by means of the exponent of their factors, leads to a more convenient method of indicating roots, which cannot be obtained algebraically, than by the sign $\sqrt{\ }$.

If it were required, for example, to find the third root of a^5, it is necessary, according to the rule given above, to divide the exponent 5 by 3; but as we cannot perform the division, we have for the quotient the fractional number $\frac{5}{3}$; and this form of the exponent indicates, that the extraction of the root is not possible in the actual state of the quantity proposed. We may, therefore, consider the two expressions

$$\sqrt[3]{a^5} \quad \text{and} \quad a^{\frac{5}{3}}$$

as equivalent.

The second, however, has this advantage over the first, that it leads directly to a more simple form, which the quantity $\sqrt[3]{a^5}$ is capable of assuming; for if we take the whole number contained in the fraction $\frac{5}{3}$, we have $1 + \frac{2}{3}$ as an equivalent exponent; consequently,

$$a^{\frac{5}{3}} = a^{1+\frac{2}{3}} = a^1 \times a^{\frac{2}{3}} \ (25);$$

from which it is evident, that the quantity $a^{\frac{5}{3}}$ is composed of two factors, the first of which is rational, and the other becomes $\sqrt[3]{a^2}$.

The same result, indeed, may be obtained from the quantity under the form $\sqrt[3]{a^5}$, by the rule given in art. 130, but the fractional exponent suggests it immediately. We shall have occasion to notice in other operations the advantages of fractional exponents.

We will merely observe for the present, that as the division of exponents, when it can be performed, answers to the extraction of roots, the indication of this division under the form of a fraction is to be regarded as the symbol of the same operation; whence,

$$\sqrt[n]{a^m} \quad \text{and} \quad a^{\frac{m}{n}}$$

are equivalent expressions.

We have rules then, which result from the assumed manner of expressing powers, which lead to particular symbols, as in art. 37, we arrived at the expression $a^0 = 1$.

133. It may be observed here, that as we divide one power by another, by subtracting the exponent of the latter from that of the former (36), fractions of a particular description may readily be reduced to new forms.

By applying the rule above referred to, we have

$$\frac{a^m}{a^n} = a^{m-n};$$

but if the exponent n of the denominator exceed the exponent m of the numerator, the exponent of the letter a in the second member will be negative.

If, for example, $m = 2$, $n = 3$, we have

$$\frac{a^2}{a^3} = a^{2-3} = a^{-1};$$

but by another method of simplifying the fraction $\frac{a^2}{a^3}$, we find it it equal to $\frac{1}{a}$, the expressions

$$\frac{1}{a} \quad \text{and} \quad a^{-1},$$

are therefore equivalent.

In general, we obtain by the rule for the exponents,

$$\frac{a^m}{a^{m+n}} = a^{m-m-n} = a^{-n},$$

and by another method

$$\frac{a^m}{a^{m+n}} = \frac{1}{a^n};$$

it follows from this, that the expressions

$$\frac{1}{a^n} \quad \text{and} \quad a^{-n},$$

are equivalent.

In fact, the sign —, which precedes the exponent n, being taken in the sense defined in art. 62, shows that the exponent in question arises from a fraction, the denominator of which contains the factor a, n times more than the numerator, which fraction is indeed $\frac{1}{a^n}$; we may, therefore, in any case which occurs, substitute one of these expressions for the other.

The quantity $\frac{a^2 b^5}{c^2 d^3}$, for example, being considered as equivalent to

$$a^2 b^5 \times \frac{1}{c^2} \times \frac{1}{d^3},$$

may be reduced to the following form,

$$a^2 b^5 c^{-2} d^{-3};$$

that is, *we may transfer to the numerator all the factors of the denominator, by giving to their exponents the sign —.*

Reciprocally, *when a quantity contains factors, which have negative exponents, we may convert them into a denominator, observing merely to give to their exponents the sign +*; thus the quantity

$$a^2 b^5 c^{-2} d^{-3},$$

becomes

$$\frac{a^2 b^5}{c^2 d^3}.$$

Of the Formation of the Powers of Compound Quantities.

134. We shall begin this section by observing, that the powers of compound quantities are denoted by including these quantities in a parenthesis, to which is annexed the exponent of the power. The expression

$$(4a^2 - 2ab + 5b^2)^3,$$

for example, denotes the third power of the quantity,

$$4a^2 - 2ab + 5b^2.$$

This power may also be expressed thus,

$$\overline{4a^2 - 2ab + 5b^2}^{\,3}.$$

135. Binomials next to simple quantities are the least complicated, yet if we undertake to form powers of these by successive multiplications, we in this way arrive only at particular results, as in art. 34, we obtained the second and third power; thus

$$(x + a)^2 = x^2 + 2ax + a^2,$$
$$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3,$$
$$(x + a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4,$$
&c.

It is not easy from this table to fix upon the law, which determines the value of the numerical coefficients. But by considering how the terms are multiplied into each other, we perceive, that the coefficients have their origin in reductions depending on the equality of the factors, which form a power. This is rendered very evident by an arrangement, which prevents these reductions taking place.

It is sufficient for this purpose to give to the several binomials

to be multiplied different second terms. If we take, for example,

$$x + a, \quad x + b, \quad x + c, \quad x + d, \text{ \&c.}$$

by performing the multiplications indicated below, and placing in the same column the terms, which involve the same power of x, we shall immediately find, that

$$\begin{array}{rllll}
(x+a)(x+b) = x^2 & + ax & + ab & & \\
& + bx & & & \\
(x+a)(x+b)(x+c) = x^3 & + ax^2 & + abx & + abc & \\
& + bx^2 & + acx & & \\
& + cx^2 & + bcx & & \\
(x+a)(x+b)(x+c(x+d) = x^4 & + ax^3 & + abx^2 & + abcx & + abcd \\
& + bx^3 & + acx^2 & + abdx & \\
& + cx^3 & + adx^2 & + acdx & \\
& + dx^2 & + bcx^2 & + bcdx & \\
& & + bdx^2 & & \\
& & + cdx^2 & &
\end{array}$$

Without carrying these products any further, we may discover the law according to which they are formed.

By supposing all the terms involving the same power of x, and placed in the same column, to form only one, as, for example,

$$a x^3 + b x^3 + c x^3 + d x^3 = (a + b + c + d)\, x^3,$$

&c.

1. *We find in each product one term more than there are units in the number of factors.*

2. *The exponent of* x *in the first term is the same as the number of factors, and goes on decreasing by unity in each of the following terms.*

3. *The greatest power of* x *has unity for its coefficient; the following, or that, whose exponent is one less, is multiplied by the sum of the second terms of the binomials; that, whose exponent is two less, is multiplied by the sum of the different products of the second terms of the binomials taken two and two; that whose exponent is three less, is multiplied by the sum of the different products of the second term of the binomials, taken three and three, and so on; in the last term, the exponent of* x*, being considered as zero* (37)*, is equal to that of the first, diminished by as many units as there are factors employed, and this term contains the product of all the second terms of the binomials.*

It is manifest, that the form of these products must be subject to the same laws, whatever be the number of factors; as may be shown by other evidence beside that from analogy.

136. It will be seen immediately, that the products, of which we are speaking, must contain the successive powers of x, from that, whose exponent is equal to the number of factors employed, to that, whose exponent is zero. To present this proposition under a general form, we shall express the number of factors by the letter m; the successive powers of x will then be denoted by

$$x^m, \quad x^{m-1}, \quad x^{m-2}, \quad \&c.$$

We shall employ the letters A, B, C, ... Y, to express the quantities, by which these powers, beginning with x^{m-1}, are to be multiplied; but as the number of terms, which depends on the particular value given to the exponent, will remain indeterminate, so long as this exponent has no particular value, we can write only the first and last terms of the expression, designating the intermediate terms by a series of points.

The formula then

$$x^m + A\,x^{m-1} + B\,x^{m-2} + Cx^{m-3} \ldots\ldots + Y,$$

represents the product of any number m of factors,

$$x + a, \; x + b, \; x + c, \; x + d, \; \&c.$$

If we multiply this by a new factor $x + l$, it becomes

$$\left.\begin{array}{l} x^{m+1} + A\,x^m + \;\; B\,x^{m-1} + \;\; C\,x^{m-2} \ldots\ldots \\ \quad\;\; + \; l\,x^m \; + l\,A\,x^{m-1} + l\,B\,x^{m-2} \ldots\ldots + l\,Y \end{array}\right\}.$$

It is evident, 1. that if A is the sum of the m second terms $a, b, c, d,$ &c. $A + l$ will be that of the $m + 1$ second terms $a, b, c, d,$ &c. l, and that consequently the expression employed to denote the coefficient will be true for the product of the degree $m + 1$, if it is true for that of the degree m.

2. If B is the sum of the products of the m quantities $a, b, c, d,$ &c. taken two and two, $B + l\,A$ will express that of the products of the $m + 1$ quantities $a, b, c, d,$ &c. l, taken also two and two; for A being the sum of the first, $l\,A$ will be that of their products by the new quantity introduced l; therefore the expression employed will be true for the degree $m + 1$, if it is for the degree m.

If C is the sum of the products of the m quantities $a, b, c, d,$ &c. taken three and three, $C + l\,B$ will be that of the products of the $m + 1$ quantities $a, b, c, d,$ &c. l, taken also three and three, since $l\,B$, from what has been said, will express the sum of the products of the first taken two and two, multiplied by the new

quantity introduced l; therefore, the expression employed will be true for the degree $m + 1$, if it is true for the degree m.

It will be seen, that this mode of reasoning may be extended to all the terms, and that the last, lY, will be the product of $m + 1$ second terms.

The propositions laid down in art. 135, being true for expressions of the fourth degree, for example, will be so, according to what has just been proved, for those of the fifth, for those of the sixth, and, being extended thus from one degree to another, they may be shown to be true generally.

It follows from this, that the product of any number whatever m, of binomial factors $x + a$, $x + b$, $x + c$, $x + d$, &c. being represented by

$$x^m + A x^{m-1} + B x^{m-2} + C x^{m-3} + \&c.$$

A will always be the sum of the m letters a, b, c, &c., B that of the products of these quantities, taken two and two, C that of the products of the quantities, taken three and three, and so on.

To comprehend the law of this expression in a single term, I take one, whose place is indeterminate, and which may be represented by $N x^{m-n}$.

This term will be the second, if we make $n = 1$, the third, if we make $n = 2$, the eleventh, if we make $n = 10$, &c. In the first case, the letter N will be the sum of the m letters a, b, c, &c. in the second, that of their products, when taken two and two; in the third, that of their products, when taken ten and ten; and in general, that of their products, taken n and n.

137. To change the products

$$(x + a)(x + b), \quad (x + a)(x + b)(x + c),$$
$$(x + a)(x + b)(x + c)(x + d), \quad \&c.$$

into powers of $x + a$, namely, into

$$(x + a)^2, \quad (x + a)^3,$$
$$(x + a)^4, \quad \&c.$$

it is only necessary to make, in the development of these products,

$$a = b, \qquad a = b = c,$$
$$a = b = c = d, \qquad \&c.$$

All the quantities, by which the same power of x is multiplied, become in this case equal; thus the coefficient of the second term, which in the product

$$(x + a)(x + b)(x + c)(x + d) \text{ is } a + b + c + d,$$

is changed into $4a$; that of the third term in the same product, which is,

$$ab + ac + ad + bc + bd + cd,$$

becomes $6a^2$. Hence it is easy to see, that the coefficients of the different powers of x will be changed into a single power of a, repeated as many times as there are terms, and distinguished by the number of factors contained in each of these terms. Thus, the coefficient N, by which the power x^{m-n} is multiplied, will, in the general development, be that power of a denoted by n, or a^n, repeated as many times, as we can form different products by taking in every possible way a number n of letters from among a number m; to find the coefficient of the term containing x^{m-n} then is reduced to finding the number of these products.

138. In order to perform the problem just mentioned, it is necessary to distinguish arrangements or *permutations* from products or *combinations*. Two letters, a and b, give only one product, but admit of two arrangements, ab and ba; three letters, a, b, c, which give only one product, admit of six arrangements (88), and so on.

To take a particular case, I will suppose the whole number of letters to be nine, namely,

$$a, b, c, d, e, f, g, h, i,$$

and that it is required to arrange them in sets of seven. It is evident, that if we take any arrangement we please, of six of these letters, $abcdef$, for example, we may join successively to it each of the three remaining letters, g, h, and i; we shall then have three arrangements of seven letters, namely,

$$abcdefg, \quad abcdefh, \quad abcdefi.$$

What has been said of a particular arrangement of six letters, is equally true of all; we conclude, therefore, that each arrangement of six letters will give three of seven, that is, as many as there remain letters, which are not employed. If, therefore, the number of arrangements of six letters be represented by P, we shall obtain the number consisting of seven letters by multiplying P by 3 or $9 - 6$. Representing the numbers 9 and 7 by m and n, and regarding P as expressing the number of arrangements, which can be furnished by m letters, taken $n - 1$ at a time, the same reasoning may be employed; we shall thus have for the number of arrangements of n letters,

$$P(m - (n - 1)), \quad \text{or} \quad P(m - n + 1).$$

This formula comprehends all the particular cases, that can occur in any question. To find, for example, the number of arrangements, that can be formed out of m letters, taken two and two, or two at a time, we make $n = 2$, which gives

$$n - 1 = 1;$$

we have then

$$P = m;$$

for P will in this case be equal to the number of letters taken one at a time; there results then from this

$$m(m - 2 + 1), \qquad \text{or} \qquad m(m - 1),$$

for the number of arrangements taken two and two.

Again, taking

$$P = m(m - 1) \qquad \text{and} \qquad n = 3,$$

we find for the number of arrangements, which m letters admit of, taken three and three,

$$m(m - 1)(m - 3 + 1) = m(m - 1)(m - 2).$$

Making

$$P = m(m - 1)(m - 2) \quad \text{and} \quad n = 4,$$

we obtain

$$m(m - 1 (m - 2)(m - 3)$$

for the number of arrangements, taken four and four. We may thus determine the number of arrangements, which may be formed from any number whatever of letters.*

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* In these arrangements it is supposed by the nature of the inquiry, that there are no repetitions of the same letter; but the theory of permutations and combinations, which is the foundation of the doctrine of chances, embraces questions in which they occur. The effect may be seen in the example we have selected, by observing, that we may write indifferently each of the 9 letters $a, b, c, d, e, f, g, h, i$, after the product of 6 letters $abcdef$. Designating, therefore, the number of arrangements, taken 6 at a time, by P, we shall have $P \times 9$ for the number of arrangements, taken 7 at a time. For the same reason, if P denote the number of arrangements of m letters, taken $n - 1$ at a time, that of their arrangements, when taken n at a time, will be Pm.

This being admitted, as the number of arrangements of m letters, taken one at a time, is evidently m, the number of arrangements, when taken 2 and 2, will be $m \times m$, or m^2, when taken 3 and 3, the number will be $m \times m \times m$, or m^3; and lastly, m^n will express the number of arrangements, when they are taken n and n.

139. Passing now from the number of arrangements of n letters, to that of their different products, we must find the number of arrangements, which the same product admits of. In order to this, it may be observed, that if in any of these arrangements, we put one of the letters in the first place, we may form of all the others as many permutations, as the product of $n - 1$ letters admits of. Let us take, for example, the product $a\,b\,c\,d\,e\,f\,g$, composed of seven letters; we may, by putting a in the first place, write this product in as many ways, as there are arrangements in the product of six letters $b\,c\,d\,e\,f\,g$; but each letter of the proposed product may be placed first. Designating then the number of arrangements, of which a product of six letters is susceptible, by Q, we shall have $Q \times 7$ for that of the arrangements of a product of seven letters. It follows from this, that if Q designate the number of arrangements, which may be formed from a product of $n - 1$ letters, $Q\,n$ will express the number of arrangements of a product of n letters.

Any particular case is readily reduced to this formula; for making $n = 2$, and observing, that when there is only one letter, $Q = 1$, we have $1 \times 2 = 2$ for the number of arrangements of a product of two letters. Again, taking $Q = 1 \times 2$ and $n = 3$, we have $1 \times 2 \times 3 = 6$ for the number of arrangements of a product of three letters; further, making $Q = 1 \times 2 \times 3$ and $n = 4$, there result $1 \times 2 \times 3 \times 4$, or 24 possible arrangements in a product of four letters, and so on[(c)].

140. What we have now said being well understood, it will be perceived, that by dividing the whole number of arrangements obtained from m letters, taken n at a time, by the number of arrangements of which the same product is susceptible, we have for a quotient the number of the different products, which are formed by taking in all possible ways n factors among these m letters. This number will, therefore, be expressed by $\frac{P(m-n+1)}{Q\,n}$;* which being considered in connexion with

* It may be observed, that if we make successively

$$n = 2, \qquad n = 3, \qquad n = 4, \text{ \&c.}$$

the formula $\frac{P\,(m-n+1)}{Q\,n}$ becomes

$$\frac{m\,(m-1)}{1\,.\,2},\ \frac{m\,(m-1)\,(m-2)}{1\,.\,2\,.\,3},\ \frac{m\,(m-1)\,(m-2)\,(m-3)}{1\,.\,2\,.\,3\,.\,4}, \text{ \&c.}$$

what was laid down in art. 137, will give $\frac{P(m-n+1)}{Qn} a^n x^{m-n}$ for the term containing x^{m-n} in the development of $(x+a)^m$.

It is evident, that the term which precedes this will be expressed by $\frac{P}{Q} a^{n-1} x^{m-n+1}$; for in going back towards the first term, the exponent of x is increased by unity, and that of a diminished by unity; moreover, P and Q are the quantities, which belong to the number $n-1$.

141. If we make $\frac{P}{Q} = M$, the two successive terms indicated above, become

$$M a^{n-1} x^{m-n+1} \text{ and } M \frac{(m-n+1)}{n} a^n x^{m-n}.$$

These results show how each term in the development of $(x+a)^m$, is formed from the preceding.

Setting out from the first term, which is x^m, we arrive at the second, by making $n = 1$; we have $M = 1$, since x^m has only unity for its coefficient; the result then is $\frac{1 \times m}{1} a x^{m-1}$, or $\frac{m}{1} a x^{m-1}$. In order to pass to the third term, we make $M = \frac{m}{1}$, and $n = 2$, and we obtain $\frac{m(m-1)}{1 \,.\, 2} a^2 x^{m-2}$. The fourth is found by supposing $M = \frac{m(m-1)}{1 \,.\, 2}$, and $n = 3$, which gives $\frac{m(m-1)(m-2)}{1 \,.\, 2 \,.\, 3} a^3 x^{m-3}$, and so on; whence we have the formula

$$(x+a)^m = x^m + \frac{m}{1} a x^{m-1} + \frac{m(m-1)}{1 \,.\, 2} a^2 x^{m-2} + \frac{m(m-1)(m-2)}{1 \,.\, 2 \,.\, 3} a^3 x^{m-3} + \&c.$$

which may be converted into this rule.

To pass from one term to the following, we multiply the numerical coefficient by the exponent of x *in the first, divide by the number, which marks the place of this term, increase by unity the exponent of* a, *and diminish by unity the exponent of* x.

Although we cannot determine the number of terms of this formula without assigning a particular value to m; yet, if we

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numbers, which express respectively, how many combinations may be made of any number m of things, taken two and two, three and three, four and four, &c.

observe the dependence of the terms upon each other, we can have no doubt respecting the law of their formation, to whatever extent the series may be carried. It will be seen, that

$$\frac{m(m-1)(m-2)\ldots.(m-n+1)}{1\ .\ 2\ .\ 3\ \ldots.\ \ n}\,a^n\,x^{m-n}$$

expresses the term, which has n terms before it.

This last formula is called the *general term* of the series

$$x^m + \frac{m}{1}\,a\,x^{m-1} + \frac{m\,(m-1)}{1\,.\,2}\,a^2\,x^{m-2} + \&c.$$

because if we make successively

$$n = 1, \quad n = 2, \quad n = 3, \ \&c.$$

it gives all the terms of this series.

142. Now, if $(x + a)^5$ be developed, according to the rule given in the preceding article; the first term being

$$x^5 \text{ or } a^0 x^5 \quad (37),$$

the second will be

$$\frac{5}{1}\,a^1\,x^4 \quad \text{or} \quad 5\,a\,x^4,$$

the third

$$\frac{5\times 4}{2}\,a^2\,x^3 \quad \text{or} \quad 10\,a^2\,x^3,$$

the fourth

$$\frac{10\times 3}{3}\,a^3\,x^2 \quad \text{or} \quad 10\,a^3\,x^2,$$

the fifth

$$\frac{10\times 2}{4}\,a^4\,x \quad \text{or} \quad 5\,a^4\,x,$$

the sixth

$$\frac{5\times 1}{5}\,a^5\,x^0 \quad \text{or} \quad a^5.$$

Here the process terminates, because in passing to the following term it would be necessary to multiply by the exponent of x in the sixth, which is zero.

This may be shown by the formula; for the seventh term, having for a numerical coefficient

$$\frac{m\,(m-1)\,(m-2)\,(m-3)\,(m-4)\,(m-5)}{1\ .\ 2\ .\ 3\ .\ 4\ .\ 5\ .\ 6},$$

contains the factor $m - 5$, which becomes $5 - 5 = 0$; and this same factor entering into each of the subsequent terms, reduces it to nothing.

Uniting the terms obtained above, we have

$$(x + a)^5 = x^5 + 5\,a\,x^4 + 10\,a^2\,x^3 + 10\,a^3\,x^2 + 5\,a^4\,x + a^5.$$

143. Any power whatever of any binomial may be developed by the formula given in art. 141. If it were required for example to form the sixth power of $2x^3 - 5a^3$, we have only to substitute in the formula the powers of $2x^3$ and $-5a^3$ respectively for those of x and a; since, if we make

$$2x^3 = x' \text{ and } -5a^3 = a',$$

we have

$$(2x^3 - 5a^3)^6 = (x' + a')^6 =$$
$$x'^6 + 6a'x'^5 + 15a'^2x'^4 + 20a'^3x'^3$$
$$+ 15a'^4x'^2 + 6a'^5x' + a'^6 \quad (141),$$

and it is only necessary to substitute for x' and a' the quantities, which these letters designate. We have then

$$(2x^3)^6 + 6(-5a^3)(2x^3)^5 + 15(-5a^3)^2(2x^3)^4$$
$$+ 20(-5a^3)^3(2x^3)^3 + 15(-5a^3)^4(2x^3)^2$$
$$+ 6(-5a^3)^5(2x^3) + (-5a^3)^6,$$

or

$$64x^{18} - 960a^3x^{15} + 6000a^6x^{12}$$
$$- 20000a^9x^9 + 37500a^{12}x^6$$
$$- 37500a^{15}x^3 + 15625a^{18}.$$

The terms produced by this development are alternately positive and negative; and it is manifest, that they will always be so, when the second term of the proposed binomial has the sign $-$.

144. The formula given in art. 141, may be so expressed as to facilitate the application of it in cases analogous to the preceding.

Since

$$x^{m-1} = \frac{x^m}{x}, \quad x^{m-2} = \frac{x^m}{x^2}, \quad x^{m-3} = \frac{x^m}{x^3}, \text{ \&c.}$$

the formula may be written

$$x^m + \frac{m}{1}\frac{a}{x}x^m + \frac{m(m-1)}{1\,.\,2}\frac{a^2}{x^2}x^m + \text{\&c.}$$

which may be reduced to

$$x^m\left\{1 + \frac{m}{1}\frac{a}{x} + \frac{m(m-1)}{1\,.\,2}\frac{a^2}{x^2} + \frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}\frac{a^3}{x^3} + \text{\&c.}\right\},$$

by insulating the common factor x^m. In applying this formula, the several steps are, *to form the series of numbers,*

$$\frac{m}{1}, \quad \frac{m-1}{2}, \quad \frac{m-2}{3}, \quad \frac{m-3}{4}, \quad \text{\&c.}$$

to multiply the first by the fraction $\frac{a}{x}$, then this product by the second

and also by the fraction $\frac{a}{x}$, *then again this last result by the third and by the fraction* $\frac{a}{x}$, *and so on; to unite all these terms, and add unity to the sum; and lastly, to multiply the whole by the factor* x^m.

In the example, $(2x^3 - 5a^3)^6$, we must write $(2x^3)^6$ in the place of x^m, and $-\frac{5a^3}{2x^3}$ in that of $\frac{a}{x}$. I shall leave the application of the formula as an exercise for the learner.†

145. We may easily reduce the development of the power of any polynomial whatever, to that of the powers of a binomial, as may be shown with respect to the trinomial $a + b + c$, the third power for instance being required.

First, we make $b + c = m$, we then obtain

$$(a + b + c)^3 = (a + m)^3 = a^3 + 3a^2m + 3am^2 + m^3;$$

substituting for m the binomial $b + c$, which it represents, we have

$$(a + b + c)^3 = a^3 + 3a^2(b + c) + 3a(b + c)^2 + (b + c)^3.$$

It only remains for us to develop the powers of the binomial $b + c$, and to perform the multiplications, which are indicated; we have then

$$\begin{array}{llll} a^3 + 3a^2b & + 3ab^2 & + b^3 & \\ \quad + 3a^2c & + 6abc & + 3b^2c & \\ & + 3ac^2 & + 3bc^2 & \\ & & + c^3. & \end{array}$$

Of the Extraction of the Roots of Compound Quantities.

146. Having explained the formation of the powers of compound quantities, I now pass to the extraction of their roots, beginning with the cube root of numbers.

In order to extract the cube root of numbers, we must first become acquainted with the cubes of numbers, consisting of only one figure; these are given in the second line of the following table;

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† The formula for the development of $(x + a)^m$ answers for all values of the exponent m, and is equally applicable to cases in which the exponent is fractional or negative. This property, which is very important, is demonstrated in a note to the last part of the Cambridge course of Mathematics on the Differential and Integral Calculus.

1	2	3	4	5	6	7	8	9
1	8	27	64	125	216	343	512	729

and the cube of 10 being 1000, no number consisting of three figures can contain the cube of a number consisting of more than one.

The cube of a number consisting of two figures is formed in a manner analogous to that, by which we arrive at the square; for if we resolve this number into tens and units, designating the first by a, and the second by b, we have

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Hence it is evident, that *the cube, or third power of a number composed of tens and units, contains four parts, namely, the cube of the tens, three times the square of the tens multiplied by the units, three times the tens multiplied by the square of the units, and the cube of the units.*

If it were required to find the third power of 47, by making $a = 4$ tens or 40, $b = 7$ units, we have

$$\begin{aligned} a^3 &= 64000 \\ 3a^2b &= 33600 \\ 3ab^2 &= 5880 \\ b^3 &= 343 \end{aligned}$$

Total, $103823 = 47 \times 47 \times 47.$

Now to go back from the cube 103823 to its root 47, we begin by observing that 64000, the cube of the 4 tens, contains no significant figure inferior to thousands; in seeking the cube of the tens therefore, we may neglect the hundreds, the tens, and the units of the number 103823. Pursuing, therefore, a method similar to that employed in extracting the square root, we separate, by a comma, the first three figures on the right; the greatest cube contained in 103 will be the cube of the tens. It is evident from the table, that this cube is 64, the root of which is 4; we therefore put 4 in the place assigned for the root. We then subtract 64 from 103; and by the side of the remainder, 39, bring down the last three figures.

103,823	47
64	48
398,23	

The whole remainder, 39823, contains still three parts of the cube, namely, three times the square of the tens multiplied by the units, or $3a^2b$, three times the tens multiplied by the square of the units, or $3ab^2$, and the cube of the units, or b^3. If the value of the product $3a^2b$ were known, we might obtain the

units b, by dividing this product by $3a^2$, which is a known quantity, the tens being now found; but, on the supposition that the product, $3a^2b$, is unknown, we readily perceive, that it can have no figure inferior to hundreds, since it contains the factor a^2, which represents the square of the tens; it must, therefore, be found in the part 398, which remains on the left of the number 39823, after the tens and units have been separated, and which contains, besides this product, the hundreds arising from the product, $3ab^2$, of the tens by the square of the units, and from the cube b^3, of the units.

If we divide 398 by 48, which is triple the square of the tens, $3a^2$ or 3×16, we obtain 8 for the quotient; but from what precedes, it appears that we ought not to adopt this figure for the units of the root sought, until we have made trial of it, by employing it in forming the three last parts of the cube, which must be contained in the remainder 39823. Making $b = 8$, we find

$$\begin{aligned} 3a^2b &= 38400 \\ 3ab^2 &= 7680 \\ b^3 &= 512 \\ \hline \text{Total,} &\quad 46592. \end{aligned}$$

As this result exceeds 39823, it is evident that the number 8 is too great for the units of the root. If we make a similar trial with 7, we find that it answers to the above conditions; 47 therefore is the root sought.

Instead of verifying the last figure of the root in the manner above described, we may raise the whole number expressed by the two figures, immediately to a cube; and this last method is generally preferred to the other. Taking the number 48 and proceeding thus, we find

$$48 \times 48 \times 48 = 110592.$$

As the result is greater than the proposed number, it is evident, that the figure 8 is too large.

147. What we have laid down in the above example may be applied to all cases, where the proposed number consists of more than three figures and less than seven. Having separated the first three figures on the right, we seek the greatest cube in the part, which remains on the left, and write its root in the usual

place; we subtract this cube from the number to which it relates, and to the remainder bring down the last three figures; separating now the tens and the units, we proceed to divide what remains on the left, by three times the square of the tens found; but before writing down the quotient as a part of the root, we verify it by raising to the cube the number consisting of the tens known, together with this figure under trial. If the result of this operation is too great, the figure for the units is to be diminished; we then proceed in the same manner with a less figure, and so on, until a root is found, the cube of which is equal to the proposed number, or is the greatest contained in this number, if it does not admit of an exact root. As we have often remainders, that are very considerable, I will here add to what has been said, a method, by which it may be soon discovered, whether or not the unit figure of the root be too small.

The cube of $a + b$, when $b = 1$, becomes that of $a + 1$,

or $$a^3 + 3a^2 + 3a + 1,$$

a quantity, which exceeds a^3, the cube of a, by

$$3a^2 + 3a + 1.$$

Hence it follows, that *whenever the remainder, after the cube root has been extracted, is less than three times the square of the root, plus three times the root, plus unity, this root is not too small.*

148. In order to extract the root of 105823817, it may be observed, that whatever be the number of figures in this root, if we resolve it into units and tens, the cube of the tens cannot enter into the last three figures on the right, and must consequently be found in 105823. But the greatest cube contained in 105823 must have more than one figure for its root; this root then may be resolved into units and tens, and, as the cube of the tens has no figure inferior to thousands, it cannot enter into the three last figures 823. If, after these are separated, there remain more than three figures on the left, we may repeat the reasoning just employed, and thus, dividing the number proposed into portions of three figures each, proceeding from right to left, and observing that the last portion may contain less than three figures, we come at length to the place occupied by the cube of the units of the highest order in the root sought.

Having thus taken the preparatory steps, we seek, by the rule given in the preceding article, the cube root of the two first por-

tions on the left, and find for the result 47; we subtract the cube of this number from the two first portions, and to the remainder 2000 bring down the following portion 817. The number 2000817 will then contain the three last parts of the cube of a number, the tens of which are 47, and the units remain to be found.

105,823,817	473
64	48
41 8,23	6627
103 823	
2 0008,17	
105 823 817	
000 000 000	

These units are therefore obtained as in the example given in the preceding article, by separating the two last figures on the right of the remainder, and dividing the part on the left by 6627, triple the square of 47. Then making trial with the quotient 3, arising from this division, by raising 473 to a cube, we obtain for the result the proposed number, since this number is a perfect cube.

The explanation, we have given, of the above example, may take the place of a general rule. If the number proposed had contained another portion, we should have continued the operation, as we have done for the third; and it is to be recollected always, that a cipher must be placed in the root, if the number to be divided on the left of the remainder happen not to contain the number used as a divisor; we should then bring down the following portion, and proceed with it, as with the preceding.

149. *Since the cube of a fraction is found by multiplying this fraction by its square, or which amounts to the same thing, by taking the cube of the numerator and that of the denominator; reversing this process, we arrive at the root, by extracting the root of the new numerator and that of the new denominator.* The cube of $\frac{5}{6}$, for example, is $\frac{125}{216}$; taking the cube root of 125 and of 216, we find $\frac{5}{6}$.

We always proceed in this way, when the numerator and denominator are perfect cubes; but when this is not the case, we may avoid the necessity of extracting the root of the denominator, by multiplying the two terms of the proposed fraction by the square of this denominator. The denominator thence arising, will be the cube of the original denominator; and it will be only necessary then to find the root of the numerator. If we have, for example, $\frac{3}{5}$, by multiplying the two terms of this fraction by 25, the square of the denominator, we obtain

$$\frac{75}{5 \times 5 \times 5}.$$

The root of the denominator is 5; while that of 75 lies between

4 and 5. Adopting 4, we have $\frac{4}{5}$ for the cube root of $\frac{3}{5}$ to within one fifth. If a greater degree of accuracy be required, we must take the approximate root of 75, by the method I shall soon proceed to explain.

If the denominator be already a perfect square, it will only be necessary to multiply the two terms of the fraction by the square root of this denominator. Thus in order to find the cube root of $\frac{4}{9}$, we multiply the two terms by 3, the square root of 9; we thus obtain

$$\frac{12}{3 \times 3 \times 3}.$$

Taking the root of the greatest cube 8, contained in 12, we have $\frac{2}{3}$ for the root sought, within one third.

150. It follows from what has been demonstrated in art. 97, that the cube root of a number, which is not a perfect cube, cannot be expressed exactly by any fraction, however great may be the denominator; it is therefore an irrational quantity, though not of the same kind with the square root; for it is very seldom that one of them can be expressed by means of the other.

151. We may obtain the approximate cube root by means of vulgar fractions. The mode of proceeding is analogous to that given for finding the square root (103); but, as it may be readily conceived, and is besides not the most eligible, I shall not stop to explain it.

A better method of employing vulgar fractions for this purpose consists in extracting the root in fractions of a given kind. Thus, if it were required to find, for example, the cube root of 22, within a fifth part of unity, observing that the cube of $\frac{1}{5}$ is $\frac{1}{125}$, we reduce 22 to $\frac{2750}{125}$; then taking the root of 2750, so far as it can be expressed in whole numbers, we have $\frac{14}{5}$, or $2\frac{4}{5}$ for the approximate root of 22.

152. It is the practice of most persons, however, in extracting the cube root of a number, by approximation, to convert this number into a decimal fraction, but it is to be observed, that this fraction must be either thousandths or millionths, or of some higher denomination; because when raised to the third power, tenths become thousandths, and thousandths millionths, and in general, *the number of decimal figures found in the cube, is triple the number contained in the root.* From this it is evident, that we must place after the proposed number three times as many ciphers, as there

are decimal places required in the root. The root is then to be extracted according to the rules already given, and the requisite number of decimal figures to be distinguished in the result.

If we would find, for example, the cube root of 327, within a hundredth part of unity, we must write six ciphers after this number, and extract the root of 327000000 according to the usual method. This is done in the following manner;

327,000,000	688
216	108
1110,00	13872
3144 32	
125 680,00	
325 660 672	
1 339 328	

Separating two figures on the right of the result for decimals, we have 6,88; but 6,89 would be more exact, because the cube of this last number, although greater than 327, approaches it more nearly than that of 6,88.

If the proposed number contain decimals already, before we proceed to extract the root, we must place on the right as many ciphers, as will be necessary to render the number of decimal figures a multiple of 3. Let there be, for example, 0,07, we must write 0,070, or 70 thousandths, which gives for a root 0,4. In order to arrive at a root exact to hundredths, we must annex three additional ciphers, which gives 0,070000. The root of the greatest cube contained in 70000 being 41, that of 0,07 becomes 0,41, to within a hundredth.

153. Hitherto I have employed the formula for binomial quantities only in the extraction of the square and cube roots of numbers; this formula leads to an analogous process for obtaining the root of any degree whatever. I shall proceed to explain this process, after offering some remarks upon the extraction of roots, the exponent of which is a divisible number.

We may find the fourth root by extracting the square root twice successively; for by taking first the square root of a fourth power, a^4, for example, we obtain the square, or a^2, the square root of which is a, or the quantity sought.

It is obvious also, that the eighth root may be obtained by extracting the square root three times successively, since the square root of a^8 is a^4, and that of a^4 is a^2, and lastly, that of a^2 is a.

In the same manner it may be shown, that all roots of a degree, designated by any of the numbers 2, 4, 8, 16, 32, &c. that is, by any power of 2, are obtained by successively extracting the square root.

Roots, the exponents of which are not prime numbers, may be reduced to others of a degree less elevated; the sixth root, for example, may be found by extracting the square and afterwards the cube root. Thus, if we take a^6 and go through this process with it, we find by the first step a^3, and by the second a; we may also take first the cube root, which gives a^2, and afterwards the square root, whence we have a, as before.

154. I now proceed to treat of the general method, which I shall apply to roots of the fifth degree. The illustration will be rendered more easy, if we take a particular example; and by comparing the different steps with the methods given, for the extraction of the square and the cube root, we shall readily perceive, in what manner we are to proceed in finding roots of any degree whatever.

Let it be required then to extract the fifth root of 231554007. Now the least number, it may be observed, consisting of 2 figures, that is 10, has in its fifth power, which is 100000, six figures; we therefore conclude, that the fifth root of the number proposed contains at least two figures; this root may then be represented by $a + b$, a denoting the tens, and b the units. The expression for the proposed number will then be

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + \&c.$$

I have not developed all the terms of this power, because it is sufficient, as will be seen immediately, that the composition of the first two be known.

Now it is evident, that as a^5, or the fifth power of the tens of this root, can have no figure, that falls below hundreds of thousands, it does not enter into the last five figures on the right of the proposed number; we, therefore, separate these five figures. If there remained more than five figures on the left, we should repeat the same reasoning, and thus separate the proposed number into portions of five figures each, proceeding from the right to the left. The last of these portions on the left, will contain the fifth power of the units of the highest order found in the root.

2315,54007	47
1024	
1291 5,4007	1280

We find, by forming the fifth powers of numbers consisting of only one figure, that 2315 lies between the fifth power of 4, or 1024, and that of five, or 3125. We take, therefore, 4 for the tens of the root sought; then subtracting the fifth power of this number, or 1024, from the first portion of the proposed number, we have for a remainder 1291. This remainder, together with the following portion, which is to be brought down, must contain $5\,a^4\,b + 10\,a^3\,b^2 +$ &c. which is left, after a^5 has been subtracted from $(a + b)^5$; but among these terms, that of the highest degree is $5\,a^4\,b$, or five times the fourth power of the tens multiplied by the units, because it has no figure, which falls below tens of thousands. In order to consider this term by itself, we separate the last four figures on the right, which make no part of it, and the number 12915, remaining on the left, will contain this term, together with the tens of thousands arising from the succeeding terms. It is obvious, therefore, that by dividing 12915 by $5\,a^4$, or five times the fourth power of the four tens already found, we shall only approximate the units. The fourth power of 4 is 256; five times this gives 1280; if we divide 12915 by 1280, we find 10 for the quotient, but we cannot put more than 9 in the place of the root, and it is even necessary, before we adopt this, to try whether the whole root 49, which we thus obtain, will not give a fifth power greater than the proposed number. We find indeed by pursuing this course, that the number 49 must be diminished by two units, and that the actual root is 47, with a remainder 2209000; for the fifth power of 47 is 229345007; that is, the exact root of the proposed number falls between 47 and 48.

If there were another portion still, we should bring it down and annex it to the remainder, resulting from the subtraction of the fifth power found as above, from the first two portions, and proceed with this whole remainder, as we did with the preceding, and so on.

After what has been said, it will be easy to apply the rules, which have been given, as well in extricating the square and cube root of fractions, as in approximating the roots of imperfect powers of these degrees.

155. We may by processes, founded on the same principles, extract the roots of literal quantities. The following example

will be sufficient to illustrate the method, which is to be employed, whatever be the degree of the root required.

We found in art. 143, the sixth power of $2x^3 - 5a^3$; we shall now extract the root of this power. The process is as follows;

$$\begin{array}{l|l}
64x^{18} - 960a^3x^{15} + 6000a^6x^{12} - 20000a^9x^9 & 2x^3 - 5a^3 \\
\qquad\qquad\qquad + 37500a^{12}x^6 - 37500a^{15}x^3 & \overline{192x^{15}} \\
\qquad\qquad\qquad\qquad\qquad\qquad + 15625a^{18} & \\
\underline{-64x^{18}\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} & \\
\text{rem.}\quad -960a^3x^{15} + \&c. &
\end{array}$$

The quantity proposed being arranged with reference to the letter x, its first term must be the sixth power of the first term of the root arranged with reference to the same letter; taking then the sixth root of $64x^{18}$, according to the rule given in art. 129, we have $2x^3$ for the first term of the root required.

If we raise this result to the sixth power, and subtract it from the proposed quantity, the remainder must necessarily commence with the second term, produced by the development of the sixth power of the two first terms of the root. But, in the expression

$$(a + b)^6 = a^6 + 6a^5b + \&c.$$

this second term is the product of six times the fifth power of the first term of the root by the second; and if we divide it by $6a^5$, the quotient will be the second term b.

We must, therefore, take six times the fifth power of the first term $2x^3$ of the root, which gives

$$6 \times 32x^{15} \text{ or } 192x^{15},$$

and divide, by this quantity, the term $-960a^3x^{15}$, which is the first term of the remainder, after the preceding operation; the quotient $-5a^3$ is the second term of the root. In order to verify it, we raise the binomial $2x^3 - 5a^3$ to the sixth power, which we find is the proposed quantity itself.

If the quantity were such as to require another term in the root, we should proceed to find, after the manner above given, a second remainder, which would begin with six times the product of the fifth power of the first two terms of the root by the third, and which consequently being divided by $6(2x^3 - 5a^3)^5$, the quotient would be this third term of the root; we should then verify it by taking the sixth power of the three terms. The same course might be pursued, whatever number of terms might remain to be found.

Of Equations with two Terms.

156. Every equation, involving only one power of the unknown quantity, combined with known quantities, may always be reduced to two terms, one of which is made up of all those, which contain the unknown quantity, united in one expression, and the other comprehends all the known quantities collected together. This has been already shown with respect to equations of the second degree, art. 105, and may be easily proved concerning those of any degree whatever.

If we have, for example, the equation

$$a^2 x^5 - a^5 b^2 = b^4 c^3 + a c x^5,$$

by bringing all the terms involving x into one member, we obtain

$$a^2 x^5 - a c x^5 = b^4 c^3 + a^5 b^2,$$

or

$$(a^2 - a c) x^5 = b^4 c^3 + a^5 b^2,$$

Now if we represent the quantities

$$a^2 - a c \text{ by } p, \; b^4 c^3 + a^5 b^2 \text{ by } q,$$

the preceding equation becomes

$$p x^5 = q;$$

freeing x^5 from the quantity, by which it is multiplied, we have

$$x^5 = \frac{q}{p};$$

whence we conclude

$$x = \sqrt[5]{\frac{q}{p}}.$$

In general, every equation with two terms being reduced to the form

$$p x^m = q,$$

gives

$$x^m = \frac{q}{p};$$

taking the root then of the degree m of each member, we have

$$x = \sqrt[m]{\frac{q}{p}}.$$

157. It must be observed, that if the exponent m is an odd number, the radical expression will have only one sign, which will be that of the original quantity (131).

When the exponent m is even, the radical expression will have

the double sign $\pm$; it will in this case be imaginary, if the quantity $\frac{q}{p}$ is negative, and the question will be absurd, like those of which we have seen examples in equations of the second degree (131).

See some examples.

The equation $$x^5 = -1024,$$
gives
$$x = \sqrt[5]{-1024} = -4,$$
the exponent 5 being an odd number.

The equation
$$x^4 = 625,$$
gives $$x = \pm\sqrt[4]{625} = \pm 5,$$
as the exponent 4 is even.

Lastly, the equation
$$x^4 = -16,$$
which gives
$$x = \pm\sqrt[4]{-16},$$
leads only to imaginary values, because while the exponent 4 is even, the quantity under the radical sign is negative.

158. I shall here notice an analytical fact, which deserves attention on account of its utility, as well in the remaining part of the present treatise, as in the *Supplement*, and which is sufficiently remarkable in itself; it is this, that all the expressions $x - a$, $x^2 - a^2$, $x^3 - a^3$, and in general $x^m - a^m$ (m being any positive whole number), are exactly divisible by $x - a$. This is obvious with respect to the first. We know that the second
$$x^2 - a^2 = (x + a)(x - a) \text{ (34)},$$
and the others may be easily decomposed by division. If we divide $x^m - a^m$ by $x - a$, we obtain for a quotient
$$x^{m-1} + a x^{m-2} + a^2 x^{m-3} + \&c.$$
the exponent of x, in each term, being less by unity than in the preceding, and that of a increasing in the same ratio. But instead of pursuing the operation through its several steps, I shall present immediately to the view the equation
$$\frac{x^m - a^m}{x - a} = x^{m-1} + a x^{m-2} + a^2 x^{m-3} \ldots + a^{m-2} x + a^{m-1},$$

which may be verified by multiplying the second member by $x - a$. It then becomes

$$\begin{array}{l} x^m + a x^{m-1} + a^2 x^{m-2} \ldots\ldots\ldots\ldots + a^{m-2} x^2 + a^{m-1} x \\ \quad - a x^{m-1} - a^2 x^{m-2} - a^3 x^{m-3} \ldots\ldots\ldots - a^{m-1} x - a^m; \end{array}$$

all the terms in the upper line, after the first, being the same, with the exception of the signs, as those preceding the last in the lower line, there only remains after reduction, $x^m - a^m$, that is, the dividend proposed.

It must be observed, that the term $a^2 x^{m-2}$, in the upper line, is necessarily followed by the term $a^3 x^{m-3}$, which is destroyed by the corresponding term in the lower line; and that, in the same manner we find, in the lower line, before the term $a^{m-1} x$, a term $- a^{m-2} x^2$, which destroys the corresponding one in the upper line. These terms are not expressed, but are supposed to be comprehended in the interval denoted by the points.

159. This leads to very important consequences, relative to the equation with two terms $x^m = \frac{q}{p}$.

If we designate by a the number, which is obtained by directly extracting the root according to the rules given in art. 154, we have

$$\frac{q}{p} = a^m \quad \text{or} \quad x^m = a^m;$$

transposing the second member we obtain

$$x^m - a^m = 0.$$

The quantity $x^m - a^m$ is divisible by $x - a$, and we have by the preceding article

$$x^m - a^m = (x - a)\,(x^{m-1} + a x^{m-2} \ldots\ldots + a^{m-2} x + a^{m-1}).$$

This last result, which vanishes when $x = a$, is also reduced to nothing, if we have

$$x^{m-1} + a x^{m-2} \ldots\ldots + a^{m-2} x + a^{m-1} = 0. \quad (116);$$

and, consequently, if there exists a value of x, which satisfies this last equation, it will satisfy also the equation proposed.

These values have with unity very simple relations, which may be discovered by making $x = ay$; then the equation $x^m - a^m = 0$ becomes

$$a^m y^m - a^m = 0, \quad \text{or} \quad y^m - 1 = 0,$$

and we obtain the values of x, by multiplying those of y by the number a.

The equation $y^m - 1 = 0$, gives in the first place

$$y^m = 1, \quad y = \sqrt[m]{1} = 1;$$

then by dividing $y^m - 1$ by $y - 1$, we have

$$y^{m-1} + y^{m-2} + y^{m-3} \ldots\ldots + y^2 + y + 1.$$

Taking this quotient for one of the members, and zero for the other, we form the equation on which the other values of y depend; and these values will, in the same manner, satisfy the equation

$$y^m - 1 = 0, \text{ or } y^m = 1,$$

that is, their power of the degree m will be unity.

Hence we infer the fact, singular at first view, that unity may have many *roots* beside itself. These roots, though imaginary, are still of frequent use in analysis. I can, however, exhibit here only those of the four first degrees, as it is only for these degrees, that we can resolve, by preceding observations, the equation

$$y^{m-1} + y^{m-2} \ldots\ldots + 1 = 0,$$

from which they are derived.

1. Let $m = 2$, we have

$$y^2 - 1 = 0,$$

whence we obtain

$$y = +1, \quad y = -1,$$

2. By making $m = 3$, we have

$$y^3 - 1 = 0,$$

whence we deduce

$$y = 1,$$

then
$$y^2 + y + 1 = 0.$$

This last equation being resolved, gives

$$y = \frac{-1 + \sqrt{-3}}{2}, \quad y = \frac{-1 - \sqrt{-3}}{2};$$

thus we have for this degree the three roots

$$y = 1, \quad y = \frac{-1 + \sqrt{-3}}{2}, \quad y = \frac{-1 - \sqrt{-3}}{2}.$$

The last two are imaginary; but if we take the cube, forming that of the numerator, by the rule given in art. 34, and observing that the square of $\sqrt{-3}$ being -3, its cube is $-3\sqrt{-3}$, we still find $y^3 = 1$, in the same manner as when we employ the root $y = 1$.

3. Taking $m = 4$, we have

$$y^4 - 1 = 0,$$

from which we deduce

$$y = 1,$$

then $$y^3 + y^2 + y + 1 = 0.$$

We are not, at present, furnished with the means of resolving this equation; but observing that

$$y^4 - 1 = (y^2 + 1)(y^2 - 1),$$

we have successively

$$y^2 - 1 = 0, \quad y^2 + 1 = 0,$$

whence

$$y = +1, \quad y = -1, \quad y = +\sqrt{-1}, \quad y = -\sqrt{-1}.$$

Two of these values only are real; and the other two imaginary.

This multiplicity of roots of unity is agreeable to a general law of equations, according to which any unknown quantity admits of as many values, as there are units in the exponent denoting the degree of the equation, by which this unknown quantity is determined; and when the question does not admit of so many real solutions, the number is completed by purely algebraic symbols, which being subjected to the operations, that are indicated, verify the equation.

Hence it follows, that there are two kinds of expressions or values for the roots of numbers; the first, which we shall term the *arithmetical determination*, is the number which is found by the methods explained in art. 154, and which answers to each particular case; the second comprehends negative values and imaginary expressions, which we shall designate by the term *algebraic determinations*, because they consist merely in the combination of algebraic signs.

Of Equations which may be resolved in the same manner as those of the Second Degree.

160. These are equations, which contain only two different powers of the unknown quantity, the exponent of one of which is double that of the other. Their general formula is

$$x^{2m} + p x^m = q,$$

p and q being known quantities.

Now if we take x^m for the unknown quantity, and make $x^m = u$, we have

$$x^{2m} = u^2,$$

whence

$$u^2 + p u = q,$$

$$u = -\tfrac{1}{2}p \pm \sqrt{q + \tfrac{1}{4}p^2} \text{ . (109);}$$

restoring x^m in the place of u, we have

$$x^m = -\tfrac{1}{2}p \pm \sqrt{q + \tfrac{1}{4}p^2},$$

an equation consisting of two terms, since the expression

$$-\tfrac{1}{2}p \pm \sqrt{q + \tfrac{1}{4}p^2},$$

as it implies only known operations, to be performed on given quantities, must be regarded as representing known quantities.

Designating the two values of this expression by a and a', we have

$$x^m = a \text{ and } x^m = a',$$

from which we obtain

$$x = \sqrt[m]{a} \text{ and } x = \sqrt[m]{a'}.$$

If the exponent m be even, instead of the two values given above, we shall have four, since each radical expression may take the sign $\pm$; then

$$x = +\sqrt[m]{a},\ x = +\sqrt[m]{a'},$$
$$x = -\sqrt[m]{a},\ x = -\sqrt[m]{a'},$$

and these four values will be real, if the quantities a and a' are positive.

All the values of x may be comprehended under one formula, by indicating directly the root of the two members of the equation

$$x^m = -\tfrac{1}{2}p \pm \sqrt{q + \tfrac{1}{4}p^2},$$

which gives

$$x = \sqrt[m]{-\tfrac{1}{2}p \pm \sqrt{q + \tfrac{1}{4}p^2}}.$$

The following question produces an equation of this kind.

161. *To resolve the number* 6 *into two such factors, that the sum of their cubes shall be* 35.

Let x be one of these factors, the other will be $\frac{6}{x}$; then taking the sum of their cubes x^3 and $\frac{216}{x^3}$, we have the equation

$$x^3 + \frac{216}{x^3} = 35,$$

which may be reduced to

$$x^6 + 216 = 35x^3,$$

or

$$x^6 - 35x^3 = -216.$$

If we consider x^3 as the unknown quantity, we obtain, by the rule given for equations of the second degree,

$$x^3 = \tfrac{35}{2} \pm \sqrt{(\tfrac{35}{2})^2 - 216}.$$

By going through the numerical calculations, which are indicated, we find

$$(\tfrac{35}{2})^2 = \tfrac{1225}{4},$$

$$\sqrt{(\tfrac{35}{2})^2 - 216} = \sqrt{\tfrac{361}{4}} = \tfrac{19}{2},$$

and consequently,

$$x^3 = \tfrac{35}{2} + \tfrac{19}{2} = \tfrac{54}{2} = 27,$$

$$x^3 = \tfrac{35}{2} - \tfrac{19}{2} = \tfrac{16}{2} = 8,$$

whence

$$x = \sqrt[3]{27} = 3,$$

$$x = \sqrt[3]{8} = 2.$$

The first value gives for the second factor $\frac{6}{3}$ or 2, while the second value presents $\frac{6}{2}$ or 3; we have, therefore, in the one case 3 and 2 for the factors sought, and in the other 2 and 3. These two solutions differ only in the order of the factors of the given number 6.

162. The equations, we have been considering, are also comprehended under the general law given in art. 159; for the values of $\sqrt[m]{a}$, $\sqrt[m]{a'}$ are to be multiplied by the roots of unity belonging to the degree denoted by the exponent m.

Applying what has been said to the equation,

$$x^6 - 35x^3 = -216,$$

we find the six following roots;

$$x = 1 \times 3, \qquad x = 1 \times 2,$$

$$x = \frac{-1 + \sqrt{-3}}{2} \times 3, \qquad x = \frac{-1 + \sqrt{-3}}{2} \times 2,$$

$$x = \frac{-1 - \sqrt{-3}}{2} \times 3, \qquad x = \frac{-1 - \sqrt{-3}}{2} \times 2,$$

of which the first two only are real.

Calculus of Radical Expressions.

163. The great number of cases, in which no exact root can be found, and the length of the operation necessary for obtaining it by approximation, have led algebraists to endeavour to perform immediately upon the quantities subjected to the radical sign, the fundamental operations, intended to be performed

upon their roots. In this way we simplify the expression as much as possible, and leave the extracting of the root, which is a more complicated process, to be performed last, when the quantities are reduced to the most simple state, which the nature of the question will allow.

The addition and subtraction of dissimilar radical quantities can take place only by means of the signs + and —. For example, the sums

$$\sqrt[3]{a} + \sqrt[5]{a}, \qquad \sqrt[3]{a} + \sqrt[3]{b},$$

and the differences

$$\sqrt[3]{a} - \sqrt[5]{a}, \qquad \sqrt[3]{a} - \sqrt[3]{b},$$

can be expressed only under their present form.

The same cannot be said of the expression

$$4a\sqrt[3]{2b} + \sqrt[3]{16a^3b} - \frac{5c}{ad}\sqrt[3]{2a^6b},$$

because the radical quantities, of which it is composed, become similar, when they are reduced to their more simple forms, according to the method explained in art. 130. First, we have

$$\sqrt[3]{16a^3b} = \sqrt[3]{8a^3 \,.\, 2b} \quad \text{or} \quad 2a\sqrt[3]{2b}$$
$$\sqrt[3]{2a^6b} = \sqrt[3]{a^6 \,.\, 2b} \quad \text{or} \quad a^2\sqrt[3]{2b};$$

the quantity, therefore, becomes

$$4a\sqrt[3]{2b} + 2a\sqrt[3]{2b} - \frac{5a^2c}{ad}\sqrt[3]{2b},$$

which gives, when reduced,

$$6a\sqrt[3]{2b} - \frac{5ac}{d}\sqrt[3]{2b} \quad \text{or} \quad (6d - 5c)\frac{a}{d}\sqrt[3]{2b}.$$

164. With respect to other operations the calculus of radical quantities depends upon the principle already referred to, namely; *that a product, consisting of several factors, is raised to any power by raising each of the factors to this power.* So also, by suppressing the radical sign, prefixed to a quantity, we raise this quantity to the power denoted by the exponent of this sign. For example, $\sqrt[7]{a}$ raised to the seventh power, is a simply, since this operation, being the reverse of that which is indicated by the sign $\sqrt[7]{\ }$, merely restores the quantity a to its original state.

According to the principles here laid down, if, for example, in the expression

$$\sqrt[7]{a} \times \sqrt[7]{b},$$

we suppress the radical signs, the result ab will be the seventh power of the above product; and taking the seventh root, we find

$$\sqrt[7]{a} \times \sqrt[7]{b} = \sqrt[7]{ab}.$$

This reasoning, which may be applied to all similar cases, shows, that in order *to multiply two radical expressions of the same degree together, we must take the product of the quantities under the radical sign, observing to place it under a sign of the same degree.*

We have by this rule

$$3\sqrt{2ab^3} \times 7\sqrt{5a^3bc} = 21\sqrt{10a^4b^4c} =$$
$$21a^2b^2\sqrt{10c};$$
$$4\sqrt{a^2-b^2} \times \sqrt{a^2+b^2} = 4\sqrt{(a^2-b^2)(a^2+b^2)} =$$
$$4\sqrt{a^4-b^4};$$
$$\sqrt[5]{\frac{2a^9-a^3b^6}{a^4-b^4}} \times \sqrt[5]{\frac{a^2b^3c^2+b^5c^2}{d^2}}$$
$$= \sqrt[5]{\frac{2a^9-a^3b^6}{a^4-b^4} \times \frac{a^2b^3c^2+b^5c^2}{d^2}}$$
$$= \sqrt[5]{\frac{a^3(2a^6-b^6)}{a^4-b^4} \times \frac{b^3c^2}{d^2}(a^2+b^2)}$$
$$= \sqrt[5]{\frac{a^3b^3c^2}{d^2} \times \frac{2a^6-b^6}{a^2-b^2}},$$

since

$$a^4-b^4 = (a^2+b^2)(a^2-b^2).$$

165. As the seventh power of the expression $\frac{\sqrt[7]{a}}{\sqrt[7]{b}}$, for example, is $\frac{a}{b}$, it will be seen, by taking the seventh root of this last result, that

$$\frac{\sqrt[7]{a}}{\sqrt[7]{b}} = \sqrt[7]{\frac{a}{b}}.$$

Hence *to divide a radical quantity by another of the same degree, we must take the quotient arising from the division of the quantities under the radical sign, recollecting to place it under a sign of the same degree.*

We find by this rule, that

$$\frac{\sqrt{6ab}}{\sqrt{3a}} = \sqrt{\frac{6ab}{3a}} = \sqrt{2b};$$

$$\frac{\sqrt{a^2-b^2}}{\sqrt{a+b}} = \sqrt{\frac{a^2-b^2}{a+b}} = \sqrt{a-b};$$

$$\frac{\sqrt[5]{a^4 b}}{\sqrt[5]{b^3 c^2}} = \sqrt[5]{\frac{a^4 b}{b^3 c^2}} = \sqrt[5]{\frac{a^4}{b^2 c^2}}.$$

166. It follows from the rule, given in art. 164, for the multiplication of radical quantities of the same degree, *that to raise a radical quantity to any power whatever, we have only to raise to this power the quantity under the radical sign, observing that the result must take the same sign;* thus to raise $\sqrt[5]{ab}$, for example, to the third power is to take the product

$$\sqrt[5]{ab} \times \sqrt[5]{ab} \times \sqrt[5]{ab},$$

and as the radical signs are all of the same degree, the quantities to which they belong, are to be multiplied together, and the radical sign to be prefixed to the product, which gives

$$\sqrt[5]{a^3 b^3}.$$

In the same manner $\sqrt[7]{a^2 b^3}$ raised to the fourth power, gives $\sqrt[7]{a^8 b^{12}}$, which may be reduced to

$$ab\sqrt[7]{ab^5},$$

by resolving $a^8 b^{12}$ into $a^7 b^7 \times ab^5$, and taking the root of the factor $a^7 b^7$ (130).

It may be observed, that *when the exponent belonging to the radical sign is divisible by that of the power to which the proposed quantity is to be raised, the operation is performed by dividing the first exponent by the second.* For example,

$$(\sqrt[6]{a})^2 = \sqrt[3]{a},$$

because $\frac{6}{2} = 3$.

Indeed $\sqrt[6]{a}$ denotes a quantity, which is six times a factor in a, and the quantity $\sqrt[3]{a}$, which is obtained by dividing 6 by 2, being only three times a factor in a, is consequently equivalent to the product of two of the first factors, and is therefore the second power of one of these factors, or of $\sqrt[6]{a}$.

The same reasoning may be applied to all similar cases, as in the following example;

$$\left(\sqrt[12]{a^2 b}\right)^3 = \sqrt[4]{a^2 b}.$$

167. If we reverse the methods given in the preceding article, we shall be furnished with rules for extracting the roots of radical quantities.

We perceive, by attending to the rule first stated, that *if the exponents of the quantities under the radical sign are divisible by that of the root required, the operation may be performed as if there were no radical sign, only it is to be observed, that the result must be placed under the original sign.*

We find, for example, that

$$\sqrt[3]{\sqrt[5]{a^6}} = \sqrt[5]{\sqrt[3]{a^6}} = \sqrt[5]{a^2}.$$

$$\sqrt[4]{\sqrt[3]{a^4 b^8}} = \sqrt[3]{\sqrt[4]{a^4 b^8}} = \sqrt[3]{a b^2}.$$

From the second rule given in the preceding article, it is evident, that *the general method for finding the root of radical quantities, is to multiply the exponent belonging to the radical sign by that of the root, which is to be extracted.*

By this last rule, we find, that

$$\sqrt[3]{\sqrt[5]{a^4}} = \sqrt[15]{a^4},$$

In fact, $\sqrt[5]{a^4}$ is a quantity, which is five times a factor in a^4 (24, 129); but the cube root of $\sqrt[5]{a^4}$, being also three times a factor in this last quantity, is found 5×3 times or 15 times a factor in the first a^4; therefore $\sqrt[3]{\sqrt[5]{a^4}} = \sqrt[15]{a^4}$. In the same manner it might be shown, that $\sqrt[5]{\sqrt[3]{a^4}} = \sqrt[15]{a^4}$.

168. Since by multiplying the exponent of a quantity under a radical sign, by any number (166), we raise the root which is indicated, to the power denoted by this number, and by multiplying also the exponent belonging to the radical sign, by the same number (167), we obtain for the result a root of a degree equal to that of the power which was before formed, it is evident, that this second operation reduces the proposed quantity back to its original state.

The expression, $\sqrt[5]{a^3}$, for example, may be changed into $\sqrt[35]{a^{21}}$, by multiplying the exponents 5 and 3 by 7; for multiplying the exponent of a^3 by 7, we have, making use of the radical sign, $\sqrt[5]{a^{21}}$, the seventh power of the proposed radical quantity, and multiplying by 7 the exponent 5 belonging to the radical sign in the expression $\sqrt[5]{a^{21}}$, we obtain the seventh root of the former result; this last process, therefore, restores the expression to its original value.

169. By this double operation, *we reduce to the same degree any number of radical quantities of different degrees, by multiplying, at the same time, the exponent belonging to each radical sign, and those of the quantities under this sign, by the product of the exponents belonging to all the other radical signs.* That the new exponents, which are thus found for the radical signs, are the same, is obvious at once, since they arise from the product of all the exponents belonging to the original radical signs; and after what has been said above, it is evident that the value of each radical quantity is the same as before.

By this rule we transform

$$\sqrt[5]{a^3 b^2} \quad \text{and} \quad \sqrt[7]{c^4 d^3},$$

into

$$\sqrt[35]{a^{21} b^{14}} \quad \text{and} \quad \sqrt[35]{c^{20} d^{15}}.$$

In the same manner the three quantities

$$\sqrt[3]{a b^2}, \quad \sqrt[5]{a^2 c^3}, \quad \sqrt[7]{b^4 c^3},$$

become respectively

$$\sqrt[105]{a^{35} b^{70}}, \quad \sqrt[105]{a^{42} c^{63}}, \quad \sqrt[105]{b^{60} c^{45}}.$$

If we meet with numbers, under the radical signs, we shall be led, in applying this rule, to raise them to the power denoted by the product of the exponents belonging to the other radical signs.

170. In the same way, we may *place under a radical sign a factor which is without one, by raising it to the power denoted by the exponent which accompanies this sign.*

We may change, for example,

$$a^2 \text{ into } \sqrt[5]{a^{10}}, \text{ and } 2a\sqrt[3]{b} \text{ into } \sqrt[3]{8a^3 b}.$$

171. After having, by the transformation explained above, reduced any radical quantities whatever, to the same degree, we may apply to them the rules, given in articles 164 and 165, for

the multiplication and division of radical quantities of the same degree.

Let there be the general expressions

$$\sqrt[m]{a^p b^q} \times \sqrt[n]{b^r c^s};$$

we change (169)

$$\sqrt[m]{a^p b^q}, \qquad \sqrt[n]{b^r c^s},$$

into $$\sqrt[mn]{a^{np} b^{nq}}, \qquad \sqrt[mn]{b^{mr} c^{ms}},$$

then by the rule given in art. 164, we have

$$\sqrt[mn]{a^{np} b^{nq}} \times \sqrt[mn]{b^{mr} c^{ms}} = \sqrt[mn]{a^{np} b^{nq+mr} c^{ms}},$$

for the product of the proposed radical quantities.

We have also by the rule, art. 165,

$$\frac{\sqrt[m]{a^p b^q}}{\sqrt[n]{b^r c^s}} = \frac{\sqrt[mn]{a^{np} b^{nq}}}{\sqrt[mn]{b^{mr} c^{ms}}} = \sqrt[mn]{\frac{a^{np} b^{nq}}{b^{mr} c^{ms}}} = \sqrt[mn]{\frac{a^{np} b^{nq-mr}}{c^{ms}}}.$$

Remarks on some peculiar cases, which occur in the Calculus of Radical Quantities.

172. The rules to which we have reduced the calculus of radical quantities, may be applied without difficulty, when the quantities employed are real. But they might lead the learner into error with regard to imaginary quantities, if they are not accompanied with some remarks upon the properties of equations with two terms.

For example, the rule laid down in art. 164, gives directly

$$\sqrt{-a} \times \sqrt{-a} = \sqrt{-a \times -a} = \sqrt{a^2};$$

and if we take $+ a$ for $\sqrt{a^2}$, we evidently come to an erroneous result, for the product $\sqrt{-a} \times \sqrt{-a}$, being the square of $\sqrt{-a}$, must be obtained by suppressing the radical sign, and is therefore equal to $- a$.

Bézout has obviated this difficulty, by observing, that when we do not know by what method the square a^2 has been formed, we must assign for its root both $+ a$ and $- a$; but when, by means of steps already taken, we know which of these two quantities multiplied by itself produced a^2, we are not allowed, in

going back to the root, to take the other quantity. This is evidently the case with respect to the expression $\sqrt{-a} \times \sqrt{-a}$; here we know, that the quantity a^2, contained under the radical sign in the expression $\sqrt{a^2}$, arises from $-a$ multiplied by $-a$; the ambiguity, therefore, is prevented, and it will be readily seen, that in taking the root, we are limited to $-a$.

The difficulty above mentioned would present itself in regard to the product $\sqrt{a} \times \sqrt{a}$, if we were not led, by the circumstance of there being no negative sign in the expression, to take immediately the positive value of $\sqrt{a^2}$. In this case, since a^2 arises from $+a$ multiplied by $+a$, its root must necessarily be $+a$.

There can be no doubt with respect to examples of the kind we have been considering; but there are cases, which can be clearly explained only by attending to the properties of equations with two terms.

173. If, for example, it were required to find the product $\sqrt[4]{a}\,\sqrt{-1}$; reducing the second of these radical expressions to the same degree with the first (169), we have

$$\sqrt[4]{a} \times \sqrt[4]{(-1)^2} = \sqrt[4]{a} \times \sqrt[4]{+1} = \sqrt[4]{a},$$

a result which is real, although it appears evident, that the quantity $\sqrt[4]{a}$ multiplied by the imaginary quantity $\sqrt{-1}$, ought to give an imaginary product. It must not be supposed, however, that the expression $\sqrt[4]{a}$ is in all respects false, but only that it is to be taken in a very peculiar sense.

In fact, $\sqrt[4]{a}$, considered algebraically, being the expression for the unknown quantity x, in the equation with two terms,

$$x^4 - a = 0,$$

admits of four different values (159); for if we make $a = \alpha^4$, by taking α to represent the numerical value of $\sqrt[4]{a}$, considered independently of its sign, or the arithmetical determination of this quantity, we have the four values

$$\alpha \times +1, \quad \alpha \times -1, \quad \alpha \times +\sqrt{-1}, \quad \alpha \times -\sqrt{-1},$$

the third of which is precisely the product proposed.

By a little attention, it will be readily perceived, whence the ambiguity of which we have been speaking, arises. The second power $+1$ of the quantity -1 under the radical sign, as it may

arise as well from $+1 \times +1$, as from -1×-1, causes the quantity $\sqrt[4]{1}$ to have two values, which are not found in $\sqrt{-1}$.

In general, the process by which the product $\sqrt[m]{a} \times \sqrt[n]{b}$ is formed, is reduced to that of raising this product to the power mn; for if we represent it by z, that is, if we make

$$\sqrt[m]{a} \times \sqrt[n]{b} = z,$$

by raising the two members of this equation, first to the power m, we have

$$a \sqrt[n]{b^m} = z^m,$$

again, raising it to the power n, we obtain

$$a^n b^m = z^{mn}.$$

This product, therefore, being determined only by means of its power of the degree mn, or by an equation of this degree with two terms, must have mn values (159). This will be perceived at once, if we reflect that the expressions $\sqrt[m]{a}$ and $\sqrt[n]{b}$, being nothing but the values of the unknown quantities x and y, in the equations with two terms,

$$x^m - a = 0, \quad y^n - b = 0,$$

and, consequently, admitting of m and of n determinations, we have, by uniting the several m determinations of x, with the several n determinations of y, mn determinations of the product required.

When we are employed upon real quantities, there is no difficulty in finding the values, because the number of those, that are real, is never more than two (157), which differ only in the sign.

174. If we use the transformation explained in art. 159, the difficulty will be confined to the roots of $+1$ and -1; for if we make $x = \alpha t$ and $y = \beta u$, α and β denoting the numerical values of $\sqrt[m]{a}$, $\sqrt[n]{b}$ considered without regard to the sign, the equations

$$x^m \mp a = 0, \qquad y^n \mp b = 0,$$

become

$$t^m \mp 1 = 0, \qquad u^n \mp 1 = 0,$$

whence

$$xy = \sqrt[m]{\pm a} \times \sqrt[n]{\pm b} = \alpha \beta t u = \alpha\beta \sqrt[m]{\pm 1} \times \sqrt[n]{\pm 1};$$

in which $\alpha\beta$ represents the product of the numbers $\sqrt[m]{a}$, $\sqrt[n]{b}$, or the

arithmetical determination of the root of the degree mn of the number $a^n b^m$.

If we would give a determinate value to the product of the radical quantities $\sqrt[m]{\pm a}$, $\sqrt[n]{\pm b}$, by fixing the degree of the radical signs, we must obtain from the equations

$$t^m \mp 1 = 0, \quad u^n \mp 1 = 0,$$

the several expressions for $\sqrt[m]{\pm 1}$, $\sqrt[n]{\pm 1}$, and combine them in a suitable manner.

To conclude, these operations are not often required, except in some very simple cases, of which the following are the principal;

1. $\sqrt{-a} \times \sqrt{-b} = \sqrt{a} \times \sqrt{b}\,(\sqrt{-1} \times \sqrt{-1})$;

I suppress the radical sign in the expression $\sqrt{-1}$, and obtain

$$\sqrt{-a} \times \sqrt{-b} = \sqrt{ab} \times -1 = -\sqrt{ab}.$$

2. $\sqrt[4]{-a} \times \sqrt[4]{-b} = \sqrt[4]{ab}\,(\sqrt[4]{-1})^2$;

I do not here multiply -1 by -1, because this would lead to the ambiguity mentioned in art. 173; but observing, that the square of the fourth root is simply the square root, we have

$$\sqrt[4]{-a} \times \sqrt[4]{-b} = \sqrt[4]{ab} \times \sqrt{-1}.$$

3. $\sqrt[6]{-a} \times \sqrt[6]{-b} = \sqrt[6]{ab} \times (\sqrt[6]{-1})^2 = \sqrt[6]{ab} \times \sqrt[3]{-1}$

$$= \sqrt[6]{ab} \times -1 = -\sqrt[6]{ab}.$$

The results will be thus found to be alternately real and imaginary.

Calculus of Fractional Exponents.

175. If we substitute in the place of the radical signs, their corresponding fractional exponents (132), and apply immediately the rules for the exponents, we shall obtain the same results, as those furnished by the methods employed in the calculus of radical quantities.

If we transform, for example,

$$\sqrt[5]{a^3 b^2}, \quad \sqrt[5]{a^3 c^2},$$

into

$$a^{\frac{3}{5}} b^{\frac{2}{5}}, \qquad a^{\frac{3}{5}} c^{\frac{2}{5}},$$

we have

$$\sqrt[5]{a^3 b^2} \times \sqrt[5]{a^3 c^2} = a^{\frac{3}{5}} b^{\frac{2}{5}} \times a^{\frac{3}{5}} c^{\frac{2}{5}} =$$

$$a^{\frac{3}{5}+\frac{3}{5}} b^{\frac{2}{5}} c^{\frac{2}{5}} = a^{\frac{6}{5}} b^{\frac{2}{5}} c^{\frac{2}{5}};$$

then, since $\frac{6}{5} = 1 + \frac{1}{5}$, and, consequently,

$$a^{\frac{6}{5}} = a^{1+\frac{1}{5}} = a \times a^{\frac{1}{5}} \quad (25),$$

and $a^{\frac{1}{5}} b^{\frac{2}{5}} c^{\frac{2}{5}}$ is equivalent to $\sqrt[5]{ab^2 c^2}$, we have

$$\sqrt[5]{a^3 b} \times \sqrt[5]{a^3 c^2} = a \sqrt[5]{ab^2 c^2},$$

a result which is not only exact, but is reduced to its most simple form.

Let there be the general example $\sqrt[m]{a^p b^q} \times \sqrt[n]{b^r c^s}$; the radical expressions here employed may be transformed into

$$a^{\frac{p}{m}} b^{\frac{q}{m}}, \quad b^{\frac{r}{n}} c^{\frac{s}{n}},$$

we then have, according to the rules for exponents, (25),

$$a^{\frac{p}{m}} b^{\frac{q}{m}} \times b^{\frac{r}{n}} c^{\frac{s}{n}} = a^{\frac{p}{m}} b^{\frac{q}{m}+\frac{r}{n}} c^{\frac{s}{n}}.$$

Now in order to add the fractions $\frac{q}{m}, \frac{r}{n}$, we must reduce them to the same denominator; and to give uniformity to the results, we must do the same with respect to the fractions $\frac{p}{m}, \frac{s}{n}$; we obtain, by this means,

$$a^{\frac{np}{mn}} b^{\frac{nq+mr}{mn}} c^{\frac{ms}{mn}};$$

and placing this result under the radical sign, we have

$$\sqrt[m]{a^p b^q} \times \sqrt[n]{b^r c^s} = \sqrt[nm]{a^{np} b^{nq+mr} c^{ms}}.$$

176. The manner of performing division is equally simple, we have for example

$$\frac{\sqrt[5]{a^3 b^2}}{\sqrt[5]{a^4 c}} = \frac{a^{\frac{3}{5}} b^{\frac{2}{5}}}{a^{\frac{4}{5}} c^{\frac{1}{5}}} = \frac{b^{\frac{2}{5}}}{a^{\frac{4}{5}-\frac{3}{5}} c^{\frac{1}{5}}} \quad (38),$$

which may be reduced to

$$\frac{b^{\frac{2}{5}}}{a^{\frac{1}{5}} c^{\frac{1}{5}}};$$

this placed under the radical sign becomes

$$\frac{\sqrt[5]{a^3 b^2}}{\sqrt[5]{a^4 c}} = \sqrt[5]{\frac{b^2}{a c}}.$$

We have in general,

$$\frac{\sqrt[m]{a^p b^q}}{\sqrt[n]{b^r c^s}} = \frac{a^{\frac{p}{m}} b^{\frac{q}{m}}}{b^{\frac{r}{n}} c^{\frac{s}{n}}} = \frac{a^{\frac{p}{m}} b^{\frac{q}{m} - \frac{r}{n}}}{c^{\frac{s}{n}}};$$

reducing the fractional exponents to the same denominator, in order to perform the subtraction, which is required, we find

$$\frac{\sqrt[m]{a^p b^q}}{\sqrt[n]{b^r c^s}} = \frac{a^{\frac{np}{mn}} b^{\frac{nq-mr}{mn}}}{c^{\frac{ms}{mn}}} = \sqrt[mn]{\frac{a^{np} b^{nq-mr}}{c^{ms}}}.$$

It is obvious, that the reduction of fractional exponents to the same denominator, answers here to the reduction of radical expressions to the same degree, and leads to precisely the same results (171).

177. It is also very evident, by the rule given in art. 127, that

$$\left(\sqrt[m]{a^p}\right)^n = \left(a^{\frac{p}{m}}\right)^n = a^{\frac{np}{m}} = \sqrt[m]{a^{np}},$$

and by the rule laid down in art. 129, that

$$\sqrt[n]{\sqrt[m]{a^p}} = \sqrt[n]{a^{\frac{p}{m}}} = a^{\frac{p}{mn}} = \sqrt[mn]{a^p}.$$

The calculus of fractional exponents affords one of the most remarkable examples of the utility of signs, when well chosen. The analogy which prevails among exponents, both fractional and entire, renders the rules, that are to be followed with respect to the latter, applicable also to the former; but a particular investigation is necessary in each case, when we use the sign $\sqrt{}$, because it has no connexion with the operation that is indicated. The further we advance in algebra the more fully shall we be convinced of the numerous advantages, which arise from the notation by exponents, introduced by Descartes.

General Theory of Equations.

178. Equations of the first and second degree are, properly speaking, the only ones, which admit of a complete solution; but there are general properties of equations of whatever degree, by which we are able to solve them, when they are numerical, and

which lead to many conclusions, of use in the higher parts of algebra. These properties relate to the particular form, which every equation is capable of assuming.

An equation in its most general form must contain all the powers of the unknown quantity, from that of the degree of the equation to the first degree, multiplied each by some known quantity, together with one term wholly known.

A general equation of the fifth degree, for example, contains all the powers of the unknown quantity, from the first to the fifth; and if there are several terms involving the same power of the unknown quantity, we must suppose them to be united in one; according to the method given for equations of the second degree, art. 108. All the terms of the equation are then to be brought into one member, as in the article above referred to; the other member will necessarily be zero; and when the first term is negative, it is rendered positive by changing the signs of all the terms of the equation.

In this way we obtain an expression similar to the following;

$$n x^5 + p x^4 + q x^3 + r x^2 + s x + t = 0,$$

in which it is to be observed, that the letters n, p, q, r, s, t, may represent negative as well as positive numbers; then dividing the whole by n, in order that the first term may have only unity for its coefficient, and making

$$\frac{p}{n} = P, \quad \frac{q}{n} = Q, \quad \frac{r}{n} = R, \quad \frac{s}{n} = S, \quad \frac{t}{u} = T,$$

we have

$$x^5 + P x^4 + Q x^3 + R x^2 + S x + T = 0.$$

In future, I shall suppose, that equations have always been prepared as above, and shall represent the general equation of any degree whatever by

$$x^n + P x^{n-1} + Q x^{n-2} \ldots\ldots + T x + U = 0.$$

The interval denoted by the points may be filled up, when the exponent n takes a determinate value.

Every quantity or expression, whether real or imaginary, which, put in the place of the unknown quantity x in an equation prepared as above, renders the first member equal to zero, and which consequently satisfies the question, is called the *root of the proposed equation;* but as the inquiry does not at present relate to powers, this acceptation of the term *root* is more general, than that, in which it has hitherto been used (90, 129).

179. Take a proposition analogous to those given in articles 116 and 159, and one which may be regarded as fundamental.

If the root of any equation whatever,

$$x^n + P x^{n-1} + Q x^{n-2} \ldots\ldots + T x + U = 0,$$

be represented by a, *the first member of this equation may be exactly divided by* x — a.

Indeed, since a is one value of x, we have, necessarily,

$$a^n + P a^{n-1} + Q a^{n-2} \ldots\ldots + T a + U = 0,$$

and, consequently,

$$U = - a^n - P a^{n-1} - Q a^{n-2} \ldots\ldots - T a,$$

so that the equation proposed is precisely the same as

$$\left.\begin{array}{r} x^n + P x^{n-1} + Q x^{n-2} \ldots\ldots + T x \\ - a^n - P a^{n-1} - Q a^{n-2} \ldots\ldots - T a \end{array}\right\} = 0,$$

which may be reduced to

$$\left.\begin{array}{r} x^n - a^n + P(x^{n-1} - a^{n-1}) + Q(x^{n-2} - a^{n-2}) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + T(x - a) \end{array}\right\} = 0.$$

As the quantities

$$x^n - a^n,\ x^{n-1} - a^{n-1},\ x^{n-2} - a^{n-2}, \ldots\ldots x - a,$$

are each divisible by $x - a$ (158), it is evident, that the first member of the proposed equation is made up of terms, all of which are divisible by this quantity, and may consequently be divided by $x - a$, as the enunciation of the proposition requires.*

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* D'Alembert has proved the same proposition in the following manner.

If we conceive the first member of the proposed equation to be divided by $x - a$, and the operation continued until all the terms involving x are exhausted, the remainder, if there be any, cannot contain x. If we represent this remainder by R, and the quotient to which we arrive by Q, we have necessarily

$$x^n + P x^{n-1} \ldots\ldots + \&c. = Q(x - a) + R.$$

Now if we substitute a in the place of x, the first member is reduced to nothing, since a is the value of x; the term $Q(x - a)$ is also nothing, because the factor $x - a$ becomes zero; we must, therefore, have $R = 0$, and it is so, independently of the substitution of a; for, as this remainder does not contain x, the substitution cannot take place, and it still preserves the value it had before.

Hence it follows, that in every case, $R = 0$, and that, consequently,

$$x^n + P x^{n-1} + Q x^{n-2}, \&c.$$

is exactly divisible by $x - a$.

180. To form the quotient we have only to substitute for the quantities

$$x^n - a^n,\ x^{n-1} - a^{n-1},\ x^{n-2} - a^{n-2}, \ldots\ldots x - a,$$

the quotients, which are obtained by dividing these quantities by $x - a$, and which are respectively

$$\begin{array}{r} x^{n-1} + a\,x^{n-2} + a^2\,x^{n-3} \ldots\ldots + a^{n-1}, \\ x^{n-2} + a\;\;x^{n-3} \ldots\ldots + a^{n-2}, \\ x^{n-3} \ldots\ldots + a^{n-3}, \\ \ldots\ldots\ldots\ldots\ldots \\ + 1. \end{array}$$

Arranging the result with reference to the powers of x, we have

$$\begin{array}{r} x^{n-1} + \;a\,x^{n-2} + \;a^2\,x^{n-3} \ldots\ldots + \quad a^{n-1}, \\ + P\,x^{n-2} + P\,a\,x^{n-3} \ldots\ldots + P\,a^{n-2}, \\ + \quad Q\,x^{n-3} \ldots\ldots + Q\,a^{n-3}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + T. \end{array}$$

181. It is evident from the rules of division simply, that if the first member of the equation,

$$x^n + P\,x^{n-1} + Q\,x^{n-2} + \&c. = 0,$$

be divided by $x - a$, the quotient obtained will be exhibited under the following form,

$$x^{n-1} + P'\,x^{n-2} + Q'\,x^{n-3} + \&c.$$

P', Q', &c. representing known quantities different from P, Q, &c. we have then

$$x^n + P\,x^{n-1} + \&c. = (x - a)\,(x^{n-1} + P'\,x^{n-2} + \&c.);$$

and according to what was observed in art. 116, the proposed equation may be verified in two ways, namely, by making

$$x - a = 0, \quad \text{or} \quad x^{n-1} + P'x^{n-2} + \&c. = 0.$$

Now if the equation

$$x^{n-1} + P'\,x^{n-2} + \&c. = 0$$

has a root b, its first member will be divisible by $x - b$; we have then

$$x^{n-1} + P'\,x^{n-2} + \&c. = (x - b)\,(x^{n-2} + P''\,x^{n-3} + \&c.),$$

and, consequently,

$$x^n + P\,x^{n-1} + \&c. = (x - a)\,(x - b)\,(x^{n-2} + P''\,x^{n-3} + \&c.);$$

the equation proposed may, therefore, be verified in three ways, namely, by making

$$x - a = 0, \text{ or } x - b = 0, \text{ or } x^{n-2} + P''x^{n-3} + \&c. = 0.$$

If the last of these equations has a root, c, its first member may still be decomposed into two factors,

$$x-c,\ x^{n-3}+P'''x^{n-4}+\&c.=0;$$

we then have

$$x^n+Px^{n-1}+\&c.$$
$$=(x-a)(x-b)(x-c)(x^{n-3}+P'''x^{n-4}+\&c.);$$

from which it is obvious, that the proposed equation may be verified in four ways, namely, by making

$$x-a=0,\ x-b=0,\ x-c=0,\ x^{n-3}+P'''x^{n-4}+\&c.=0.$$

Pursuing the same reasoning, we obtain successively factors of the degrees

$$n-4,\quad n-5,\quad n-6,\ \&c.;$$

and if each of these factors being put equal to zero, is susceptible of a root, the first member of the proposed equation is reduced to the form

$$(x-a)(x-b)(x-c)(x-d)\ldots\ldots(x-l),$$

that is, it is decomposed into as many factors of the first degree, as there are units in the exponent, n, which denotes the degree of the equation.

The equation

$$x^n+Px^{n-1}+\&c.=0,$$

may be verified in n ways, namely, by making

$$x-a=0, \text{ or } x-b=0, \text{ or } x-c=0, \text{ or } x-d=0,$$

or lastly, $\qquad x-l=0.$

It is necessary to observe, that these equations are to be regarded as true only when taken one after the other, and there arise manifest contradictions from the supposition, that they are true at the same time. In fact, from the equation $x-a=0$, we obtain $x=a$, while $x-b=0$ gives $x=b$, results, which are inconsistent, when a and b are unequal quantities.

182. If the first member of the proposed equation,

$$x^n+Px^{n-1}+\&c.=0,$$

be decomposed into n factors of the first degree,

$$x-a,\ x-b,\ x-c,\ x-d,\ldots\ldots x-l,$$

it cannot be divided by any other expression of this degree. Indeed, if it were possible to divide it by a binomial $x-\alpha$, different from the former ones, we should have

$$x^n+Px^{n-1}+\&c.=(x-\alpha)(x^{n-1}+px^{n-2}+\&c.)$$

and, consequently,

$$(x-a)(x-b)(x-c)(x-d)\dots\dots(x-l)$$
$$=(x-\alpha)(x^{n-1}+px^{n-2}+\&c.);$$

now by changing x into α, this becomes

$$(\alpha-a)(\alpha-b)(\alpha-c)(\alpha-d)\dots\dots(\alpha-l)$$
$$=(\alpha-\alpha)(\alpha^{n-1}+p\alpha^{n-2}+\&c.)$$

The second member vanishes by means of the factor $\alpha-\alpha$, which is nothing; this is not the case with respect to the first, which is the product of factors, all of which are different from zero, so long as α differs from the several roots $a, b, c, d \dots l$. The supposition we have made then is not true; therefore, *an equation of any degree whatever does not admit of more binomial divisors of the first degree, than there are units in the exponent denoting its degree, and consequently, cannot have a greater number of roots.**

183. An equation regarded as the product of a number of factors,

$$x-a,\ x-b,\ x-c,\ x-d,\ \&c.,$$

equal to the exponent of its degree, may take the form of the product exhibited in art. 135, with this modification, that the terms will be alternately positive and negative.

If we take four factors, for example, we have

$$\begin{array}{llll} x^4 - ax^3 & + abx^2 & - abcx & + abcd = 0. \\ \quad\ - bx^3 & + acx^2 & - abdx & \\ \quad\ - cx^3 & + adx^2 & - acdx & \\ \quad\ - dx^3 & + bcx^2 & - bcdx & \\ & + bdx^2 & & \\ & + cdx^2 & & \end{array}$$

The second terms of the binomials $x-a$, $x-b$, $x-c$, &c. being the roots of the equation, taken with the contrary sign, the properties enumerated in art. 135, and proved generally in art. 136, will, in the present case, be as follows,

The coefficient of the second term, taken with the contrary sign, will be the sum of the roots;

The coefficient of the third term will be the sum of the products of the roots, taken two and two;

The coefficient of the fourth term, taken with the contrary sign, will be the sum of the products of the roots, multiplied three and

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* This demonstration is taken from the *Annales de Mathématiques* published by M. Gergonne. See vol. iv, pp. 209, 210, note.

three, and so on, the signs of the coefficients of the even terms being changed;

The last term, subject also to this law, *will be the product of all the roots.*

Making, for example, the product of the three factors

$$x - 5,\ x + 4,\ x + 3,$$

equal to zero, we form the equation

$$x^3 + 2x^2 - 22x - 60 = 0,$$

the roots of which are

$$+5, -4, -3;$$

we have for their sum

$$5 - 4 - 3 = -2;$$

for the sum of their products, taken two and two,

$$+5 \times -4 + 5 \times -3 - 4 \times -3 = -20 - 15 + 12 = -23,$$

and for the product of the three roots,

$$+5 \times -4 \times -3 = 60.$$

In this way we form the coefficients, 2, $-$ 23, $-$ 60, changing the signs of those for the second and fourth terms.

If we make the product of the factors

$$x - 2,\ x - 3, \text{ and } x + 5,$$

equal to zero, the equation thence arising

$$x^3 - 19x + 30 = 0,$$

as it has no term involving x^2, the power immediately inferior to that of the first term, *wants the second term;* and the reason is, that the sum of the roots, which, taken with the contrary sign, forms the coefficient of this term, is here

$$2 + 3 - 5,$$

or zero, or in other words, the sum of the positive roots is equal to that of the negative.*

184. We have proved (182), that an equation, considered as arising from the product of several simple factors, or factors of the first degree, can contain only as many of these factors, as there are units in the exponent n denoting the degree of this equation; but if we combine these factors two and two, we form quantities of the second degree, which will also be factors of the proposed equation, the number of which will be expressed by

$$\frac{n(n-1)}{1 \cdot 2} \quad (140).$$

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* See note at the end of this treatise.

For example, the first member of the equation

$$\begin{array}{llll} x^4 - ax^3 & + abx^2 & - abcx & + abcd = 0 \\ - bx^3 & + acx^2 & - abdx & \\ - cx^3 & + adx^2 & - acdx & \\ - dx^3 & + bcx^2 & - bcdx & \\ & + bdx^2 & & \\ & + cdx^2 & & \end{array}$$

being the product of

$$(x - a) \times (x - b) \times (x - c) \times (x - d),$$

may be decomposed into factors of the second degree, in the six following ways;

$$\begin{array}{l} (x - a)\,(x - b) \times (x - c)\,(x - d) \\ (x - a)\,(x - c) \times (x - b)\,(x - d) \\ (x - a)\,(x - d) \times (x - b)\,(x - c) \\ (x - b)\,(x - c) \times (x - a)\,(x - d) \\ (x - b)\,(x - d) \times (x - a)\,(x - c) \\ (x - c)\,(x - d) \times (x - a)\,(x - b); \end{array}$$

whence it appears, that an equation of the fourth degree may have six divisors of the second.

By combining the simple factors three and three, we form quantities of the third degree for divisors of the proposed equation; for an equation of the degree n the number will be

$$\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3},$$

and so on.

Of Elimination among Equations exceeding the First Degree.

185. The rule given in art. 78, or the method pointed out in art. 84, is sufficient, in all cases, for eliminating in two equations an unknown quantity, which does not exceed the first degree, whatever may be the degree of the others; and the rule of art. 78, is applicable, even when the unknown quantity is of the first degree in only one of the proposed equations.

If we have, for example, the equations

$$ax^2 + bxy + cy^2 = m^2,$$
$$x^2 + xy = n^2,$$

taking, in the second, the value of y, which will be

$$y = \frac{n^2 - x^2}{x},$$

and substituting this value and its square, in the place of y and y^2 in the first equation, we obtain a result involving only x.

186. If both of the proposed equations involved the second power of each of the two unknown quantities, the above method could be applied in resolving only one of the equations, either with respect to x or y.

Let there be, for example, the equations

$$\begin{aligned} ax^2 + bxy + cy^2 &= m^2, \\ x^2 + y^2 &= n^2; \end{aligned}$$

the second gives

$$y = \pm \sqrt{n^2 - x^2};$$

Substituting this value of y, and its square in the first, we obtain

$$ax^2 \pm bx\sqrt{n^2 - x^2} + c(n^2 - x^2) = m^2.$$

Our purpose appears to be answered, since we have arrived at a result, which does not involve the unknown quantity y, but we are unable to resolve the equation containing x, without reducing it to a rational form, by making the radical sign, under which the unknown quantity is found, to disappear.

It will be readily seen, that if this radical expression stood alone in one member, we might make the radical sign to disappear by raising this member to a square. Collecting together all the rational terms then in one member, by transposing the terms $\pm bx\sqrt{n^2 - x^2}$ and m^2, we have

$$ax^2 + c(n^2 - x^2) - m^2 = \mp bx\sqrt{n - x^2};$$

taking the square of each member, we form the equation

$$\left.\begin{array}{l} a^2x^4 + c^2(n^2 - x^2)^2 + m^4 \\ + 2acx^2(n^2 - x^2) - 2am^2x^2 - 2cm^2(n^2 - x^2) \end{array}\right\} = b^2x^2(n^2 - x^2),$$

which contains no radical expression.

The method, we have just employed for making the radical sign to disappear, deserves attention, on account of the frequent occasion we have to apply it; it consists in *insulating the quantity found under the radical sign, and then raising the two members of the proposed equation to the power denoted by the degree of this sign.*

187. The complicated nature of this process, which increases in proportion to the number of radical expressions, added to the difficulty of resolving one of the proposed equations with reference to one of the unknown quantities, a difficulty, which is often insurmountable in the present state of algebra, has led those, who have cultivated this science, to seek a method of effecting the

elimination without this; so that the resolution of the equations shall be the last of the operations required for the solution of the problem.

In order to render the operation more simple, we reduce equations with two unknown quantities to the form of equations with only one, by presenting only that, which we wish to eliminate. If we have, for example,

$$x^2 + axy + bx = cy^2 + dy + e,$$

we bring all the terms into one member, and arrange them with reference to x; the equation then becomes

$$x^2 + (ay + b)\,x - cy^2 - dy - e = 0;$$

abridging this, by making

$$ay + b = P, \quad -cy^2 - dy - e = Q,$$

we have

$$x^2 + Px + Q = 0.$$

The general equation of the degree m with two unknown quantities must contain all the powers of x and y, which do not exceed this degree, as well as those products, in which the sum of the exponents of x and y does not exceed m; this equation then may be represented thus;

$$x^m + (a+by)x^{m-1} + (c+dy+ey^2)x^{m-2} + (f+gy+hy^2+ky^3)x^{m-3}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+(p+qy+ry^2\ldots+uy^{m-1})x + p'+q'y+r'y^2\ldots+v'y^m = 0.$$

No coefficient is assigned to x^m in this equation, because we may always, by division, free any term of an equation we please, from the number, by which it is multiplied. Now if we make

$$a+by = P, \quad c+dy+ey^2 = Q, \quad f+gy+hy^2+ky^3 = R,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$p+qy\ldots\ldots+uy^{m-1} = T, \quad p'+q'y\ldots\ldots+v'y^m = U,$$

the above equation takes the following form,

$$x^m + Px^{m-1} + Qx^{m-2} + Rx^{m-3}\ldots\ldots + Tx + U = 0.$$

188. It should be observed, that we may immediately eliminate x in the two equations of the second degree,

$$x^2 + Px + Q = 0, \quad x^2 + P'x + Q' = 0,$$

by subtracting the second from the first. This operation gives

$$(P - P')\,x + Q - Q' = 0,$$

whence
$$x = -\frac{Q - Q'}{P - P'};$$

substituting this value in one of the two proposed equations, the first for example, we find

$$\frac{(Q-Q')^2}{(P-P')^2}-\frac{P(Q-Q')}{P-P'}+Q=0;$$

making the denominators to disappear, we have

$$(Q-Q')^2-P\,(P-P')\,(Q-Q')+Q\,(P-P')^2=0,$$

then developing the two last terms, and making the reduction

$$(Q-Q')^2+(P-P')\,(P\,Q'-Q\,P')=0.$$

We have then only to substitute for P, Q, P', and Q', the particular values, which answer to the case under consideration.

189. Before proceeding further, I shall show, how we may determine, whether the value of any one of the unknown quantities satisfies at the same time the two equations proposed. In order to make this more clear, I shall take a particular example; the reasoning employed will, however, be of a general nature.

Let there be the equations

$$x^3+3x^2y+3xy^2-98=0\ldots\ldots(1),$$
$$x^2+4xy-2y^2-10=0\ldots\ldots(2),$$

which we shall suppose furnished by a question, that gives $y=3$.

In order to verify this supposition, we must substitute 3 in the place of y, in the proposed equation; we have then

$$x^3+9x^2+27x-98=0\ldots\ldots(\text{a}),$$
$$x^2+12x-28=0\ldots\ldots(\text{b}),$$

equations, which must present the same value of x, if that, which has been assigned to y, be correct. If the value of x be represented by α, the equation (a) and the equation (b) will, according to what has been proved in art. 179, both of them be divisible by $x-\alpha$; they must, therefore, have a common divisor, of which $x-\alpha$ forms a part; and in fact, we find for this common divisor $x-2$ (48); we have therefore $\alpha=2$. Thus the value $y=3$ fulfils the conditions of the question, and corresponds to $x=2$.

If there remained any doubt, whether or not the common divisor of the equations (a) and (b) must give the value of x, we might remove it, by observing, that these equations reduce themselves to

$$(x^2+11x+49)\,(x-2)=0,$$
$$(x+14)\,(x-2)=0,$$

from which it is evident, that they are verified by putting 2 in the place of x.

190. The method I have just explained, for finding the value of x, when that of y is known, may be employed immediately in the elimination of x.

Indeed, if we take the equations (1) and (2), and go through the process necessary for determining whether they have a common divisor involving x, instead of finding one, we arrive at a remainder, which contains only the unknown quantity y and numbers, that are given; and it is evident, that if we put in the place of y its value 3, this remainder will vanish, since by the same substitution, the equations (1) and (2), become the equations (a) and (b), which have a common divisor. Forming an equation, therefore, by taking this remainder and zero for the two members, we express the condition, which the values of y must fulfil, in order that the two given equations may admit, at the same time, of the same value for x.

The adjoining table presents the several steps of the operation relative to the equations,

$$x^3 + 3x^2y + 3xy^2 - 98 = 0,$$
$$x^2 + 4xy - 2y^2 - 10 = 0,$$

on which we have been employed in the preceding article. We find for the last divisor,

$$(9y^2 + 10)x - 2y^3 - 10y - 98;$$

and the remainder, being taken equal to zero, gives

$$43y^6 + 345y^4 - 1960y^3 + 750y^2 - 2940y - 4302 = 0,$$

an equation, which admits, besides the value $y = 3$ given above, of all the other values of y, of which the question proposed is susceptible.

The remainder above mentioned being destroyed, that preceding the last becomes the common divisor of the equations proposed; and being put into an equation, gives the value of x when that of y is introduced. Knowing, for example, that $y = 3$, we substitute this value in the quantity

$$(9y^2 + 10)x - 2y^3 - 10y - 98;$$

then taking the result for one member, and zero for the other, we have the equation of the first degree

$$91x - 182 = 0, \quad \text{or} \quad x = 2.$$

191. The operation to which the above equations have been subjected, furnishes occasion for several important remarks. First, it may happen that the value of y reduces the remainder preceding the last to nothing; in this case, the next higher remainder, or that which involves the second power of x, becomes the common divisor of the two proposed equations. Introducing then into this the value of y, and putting it equal to zero,

$$\begin{array}{r|l}
x^3+3x^2y+3y^2x-98 & x^2+4xy-2y^2-10 \\ \hline
-x^3-4x^2y+2y^2x+10x & x-y \\ \hline
-x^2y+5y^2x+10x-98 & \\
+x^2y+4y^2x-2y^3-10y & \\ \hline
\text{1st rem.}\ldots\ldots+(9y^2+10)x-2y^3-10y-98 &
\end{array}$$

$$\begin{array}{r|l}
x^2+4xy-2y^2-10 & (9y^2+10)x-2y^3-10y-98 \\ \hline
\text{or rather } (9y^2+10)x^2+36xy^3-18y^4-110y^2-100 & (9y^2+10)x-2y^3-10y-98 \\
+40xy & \\ \hline
-(9y^2+10)x^2+2xy^3+98x & x+38y^3+50y+98 \\
+10xy & \\ \hline
+38xy^3-18y^4-110y^2-100 & \\
+50xy & \\
+98x &
\end{array}$$

$$\begin{array}{ll}
\text{or rather} & (38y^3+50y+98)(9y^2+10)x-162y^6-1170y^4-2000y^2-1000 \\
& -(38y^3+50y+98)(9y^2+10)x+76y^6+480y^4+3920y^3+500y^2+5880y+9604 \\ \hline
\text{2d rem.} \ldots & \ldots\ldots\ldots\ldots\ldots\ldots-86y^6-690y^4+3920y^3-1500y^2+5880y+8604
\end{array}$$

Putting this remainder equal to zero, then dividing all its terms by 2, and changing the signs in order to make the first term positive, we have

$$43y^6+345y^4-1960y^3+750y^2-2940y-4302=0,$$

we have an equation of the second degree, involving only x, the two values of which will correspond to the known value of y. If this value still reduce to nothing the remainder of the second degree, we must go back to the preceding, or that into which the third power of x enters, because this, in the case under consideration, becomes the common divisor of the two proposed equations; and the value of y will correspond to the three values of x. In general, we must go back until we arrive at a remainder, which is not destroyed by substituting the value of y.

It may sometimes happen, that there is no remainder, or that the remainder contains only known quantities.

In the first case, the two equations have a common divisor independently of any determination of y; they assume then the following form,

$$P \times D = 0, \quad Q \times D = 0,$$

D being the common divisor. It is evident, that we satisfy both the equations at the same time, by making in the first place $D = 0$; and this equation will enable us to determine one of the unknown quantities by means of the other, when the factor D contains both; but if it contains only given quantities and x, this unknown quantity will be determinate, and the other will remain wholly indeterminate. With respect to the factors, which do not contain x, they are found by what is laid down in art. 50.

Next, if we make at the same time

$$P = 0, \quad Q = 0,$$

we have still two equations, which will furnish solutions of the question proposed.

Let there be, for example,

$$(a\,x + b\,y - c)\,(mx + ny - d) = 0,$$
$$(a'x + b'y - c')\,(mx + ny - d) = 0;$$

by supposing, first, the second factor, common to the two equations, to be nothing, we have with respect to the unknown quantities x and y only the equation

$$mx + ny - d = 0,$$

and in this view the question will be indeterminate; but if we suppress this factor, we are furnished with the equations

$$a\,x + b\,y - c = 0, \quad a'x + b'y - c' = 0,$$

or $$a\,x + b\,y = c, \quad a'x + b'y = c';$$

and in this case the question will be determinate, since we have as many equations as unknown quantities.

When the remainder contains only given quantities, the two proposed equations are contradictory; for the common divisor, by which it is shown that they may both be true at the same time, cannot exist, except by a condition which can never be fulfilled.(D) This case corresponds to that mentioned in art. 68, relative to equations of the first degree.*

192. If then we have any two equations,

$$x^m + Px^{m-1} + Qx^{m-2} + Rx^{m-3} \ldots\ldots + Tx + U = 0,$$
$$x^n + P'x^{n-1} + Q'x^{n-2} + R'x^{n-3} \ldots\ldots + Y'x + Z' = 0,$$

where the second unknown quantity, y, is involved in the coefficients, P, Q, &c. P', Q', &c. in seeking the greatest common divisor of their first members, we resolve them into other more simple expressions, or come to a remainder independent of x, which must be made equal to zero.

This remainder will form the *final equation* of the question proposed, if it does not contain factors foreign to this question; but it very often begins with polynomials involving y, by which the highest power of x, in the several quantities, that have been successively employed as divisors, is multiplied, and we arrive at a result more complicated than that which is sought should be. In order to avoid being led into error with respect to the values of y arising from these factors, the idea, which first presents itself, is, to substitute immediately in the equations proposed each of the values furnished by the equation involving y only; for all the values, which give a common divisor to these equations, necessarily belong to the question, and the others must be excluded. It will be perceived also, that the final equation will

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* It will be readily perceived, by what precedes, that the problem for obtaining the final equation from two equations with two unknown quantities, is, in general, determinate; but the same final equation answers to an infinite variety of systems of equations with two unknown quantities. Reversing the process, by which the greatest common divisor of two quantities is obtained, we may form these systems at pleasure; but as this inquiry relates to what would be of little use in the elementary parts of mathematics, and would lead me into tedious details, I shall not pursue it here. Researches of this nature must be left to the sagacity of the intelligent reader, who will not fail, as occasion offers, of arriving at a satisfactory result.

become incomplete, if we suppress in the operation any factor involving y; but all these circumstances together occasion some inconvenience in the application of the above method,* and lead me to prefer the method given by Euler, which I shall explain in the following article.

193. Let there be the equations

$$x^3 + Px^2 + Qx + R = 0,$$
$$x^4 + P'x^3 + Q'x^3 + R'x + S' = 0;$$

representing by $x - \alpha$ the factor, which must be common to both, when y is determinate in a proper sense, we may consider the first as the product of $x - \alpha$ by the factor of the second degree, $x^2 + px + q$, and the second as the product of $x - \alpha$ by the factor of the third degree $x^3 + p'x^2 + q'x + r'$, p and q, p', q' and r' being indeterminate coefficients. We have then

$$x^3 + Px^2 + Qx + R = (x - \alpha)(x^2 + px + q),$$
$$x^4 + P'x^3 + Q'x^2 + R'x + S' = (x - \alpha)(x^3 + p'x^2 + q'x + r'),$$

Exterminating the binomial $(x - \alpha)$, in the same manner as an unknown quantity of the first degree (84), we find

$$(x^3 + Px^2 + Qx + R)(x^3 + p'x^2 + q'x + r') =$$
$$(x^4 + P'x^3 + Q'x^2 + R'x + S')(x^2 + px + q);$$

a result, which must verify itself without any particular value being assigned to x; this cannot take place, however, unless the first member be composed of the same terms as the second; we must, therefore, after performing the multiplications, which are indicated, put the coefficients belonging to each power of x in one member, respectively equal to those belonging to the same power in the other. In this way we obtain the following equations;

$$P + p' = P' + p \qquad Rp' + Qq' + Pr' = S' + R'p + Q'q$$
$$Q + Pp' + q' = Q' + P'p + q \qquad Rq' + Qr' = S'p + R'q$$
$$R + Qp' + Pq' + r' = R' + Q'p + P'q \qquad Rr' = S'q.$$

As we have here six equations, and only five indeterminate quantities, namely, p, q, p', q', and r', all of which are of the first

* On this subject see a memoir of M. Bret, in the 15th number of *Journal de l'Ecole Polytechnique*, also one of M. Lefébure, 3d number, vol. ii. of the *Correspondance* of the same school.

degree, these quantities may be exterminated; we shall thus arrive at an equation, which, involving only the quantities P, Q, R, P', Q', R', and S', will express a condition necessarily implied in the conditions of the question, and which, consequently, will be the final equation in y.*

Should this equation be identical, it follows, that the proposed equations have at least one factor of the form $x - \alpha$, whatever y may be; on the contrary, if the final equation contain only known quantities, the proposed equations are contradictory.

When the final equation takes place, we obtain the factor $x - \alpha$ by dividing the first of the proposed equations by the polynomial $x^2 + px + q$; we find for the quotient

$$x + P - p,$$

and neglect the remainder, because it must necessarily be reduced to nothing, when we substitute in the place of y a value obtained from the final equation. Putting the above quotient equal to zero, we find

$$x = p - P,$$

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* The method of Euler, explained here, amounts to multiplying each of the proposed equations by a factor, the coefficients of which are indeterminate, putting the products equal, and disposing the coefficients in such a manner, that the terms containing the unknown quantity destroy each other. In this form it is presented in his *Introduction to the analysis of infinites.* The exponent, which denotes the degree of the products, being designated by k, that of the factors is $k - m$ for the equation of the degree m, and $k - n$ for that of the degree n. The first term of each of these factors, having unity for a coefficient, the one contains $k - m$ indeterminate coefficients, and the other $k - n$. The sum of the products contains a number k of terms involving x; but it is necessary to destroy $k - 1$ terms only, because that, which contains the highest power of x, vanishes of itself. It follows from this, that the whole number $2k - m - n$ of indeterminate coefficients must be equal to $k - 1$, and consequently $k = m + n - 1$; we must, therefore, multiply the equation of the degree m by a factor of the degree $n - 1$, that of the degree n by a factor of the degree $m - 1$, and put the products equal, term to term, a method similar to that given in the text. It may be observed, that this former method of Euler contains the germ of that developed by Bézout in his *Théorie des Equations Algébriques.*

and this value of x will be known, or at least will be expressed by means of y, if we substitute for p its value deduced from the equations of the first degree, formed above.

This expression assumes, in general, a fractional form, so that we have $x = \frac{M}{N}$, or $Nx - M = 0$; and it may be seen in this case, that the values of y, which would cause M and N to vanish at the same time, would verify the preceding equation independently of x; this takes place in consequence of the fact, that by means of these values, the proposed equations would acquire a common factor of a degree above the first. It would not be difficult to go back to the immediate conditions in which this circumstance is implied; but the limits I have prescribed to myself in the present treatise do not permit me to enter into details of this kind.

194. Now let there be the equations

$$x^2 + Px + Q = 0, \quad x^2 + P'x + Q' = 0;$$

the factors, by which $x - a$ is multiplied, will be here of the first degree, or $x + p$ and $x + p'$ simply; in this case,

$$R = 0,\ R' = 0,\ S' = 0,\ q = 0,\ q' = 0,\ r' = 0,$$

and we have

$$\left.\begin{aligned} P + p' &= P' + p \\ Q + Pp' &= Q' + P'p \\ Qp' &= Q'p \end{aligned}\right\} \text{ or } \left\{\begin{aligned} p - p' &= P - P' \\ P'p - Pp' &= Q - Q' \\ Q'p - Qp' &= 0. \end{aligned}\right.$$

From the first two equations we obtain

$$p = \frac{(P - P')P - (Q - Q')}{P - P'};$$

$$p' = \frac{(P - P')P' - (Q - Q')}{P - P'}.$$

Substituting these values in the third, we have

$$(P - P')Q'P - (Q - Q')Q' = (P - P')P'Q - (Q - Q')Q$$

or $$(P - P')(PQ' - QP') + (Q - Q')^2 = 0.$$

Now if in the equation

$$x = p - P;$$

we put, in the place of p, its value found above, we have

$$x = -\frac{Q - Q'}{P - P'}.$$

195. In order to aid the learner, I shall indicate the operations necessary for eliminating x in the two equations

$$x^3 + Px^2 + Qx + R = 0, \quad x^3 + P'x^2 + Q'x + R' = 0.$$

In this case, we have

$$S' = 0, \quad r' = 0 \quad (193),$$

and are furnished with these five equations;

$$\begin{aligned} P + p' &= P' + p, \\ Q + Pp' + q' &= Q' + P'p + q, \\ R + Qp' + Pq' &= R' + Q'p + P'q, \\ Rp' + Qq' &= R'p + Q'q, \\ Rq' &= R'q, \end{aligned}$$

which may take the following form,

$$\begin{aligned} p - p' &= P - P', \\ P'p - Pp' + q - q' &= Q - Q', \\ Q'p - Qp' + P'q - Pq' &= R - R', \\ R'p - Rp' + Q'q - Qq' &= 0, \\ R'q - Rq' &= 0. \end{aligned}$$

We may, by the rules given in art. 88, obtain immediately from any four of these equations, the values of the unknown quantities p, p', q and q'; but the simple form, under which the first and the last of the equations are presented, enables us to arrive at the result, by a more expeditious method. In order to abridge the expressions, we make

$$P - P' = e, \quad Q - Q' = e', \quad R - R' = e'';$$

and proceed to deduce from the first and last of the proposed equations,

$$p' = p - e, \quad q' = \frac{R'q}{R};$$

then substituting these values in the three others, and making the denominator R to disappear, we have

$$\begin{aligned} (P' - P)\,Rp + (R - R')\,q &= R\,(e' - Pe) \ldots (a), \\ (Q' - Q)\,Rp + (RP' - PR')\,q &= R\,(e'' - Qe) \ldots (b), \\ (R' - R)\,Rp + (RQ' - QR')\,q &= -R^2\,e \ldots\ldots (c). \end{aligned}$$

If now we obtain, from the equations (a) and (b), the values of p and q (88), and suppress the factor R, which will be common to the numerators and the denominator, we have

$$p = \frac{(e' - Pe)\,(RP' - PR') - (R - R')\,(e'' - Qe)}{(P' - P)\,\;(RP' - PR') - (R - R')\,(Q' - Q)},$$

$$q = \frac{(P' - P)\,(e'' - Qe)\,R - R\,(e' - Pe)\,(Q' - Q)}{(P' - P)\,(RP' - PR') - (R - R')\,\;(Q' - Q)};$$

putting these values in the equation (c), we obtain a final equation, divisible by R, and which may be reduced to

$$(R' - R)[(e' - Pe)(RP' - PR') - (R - R')(e'' - Qe)]$$
$$+ (RQ' - QR')[(P' - P)(e'' - Qe) - (e' - Pe)(Q' - Q)]$$
$$= - Re[(P' - P)(RP' - PR') - (R - R')(Q' - Q)];$$

it only remains then to substitute for the letters e, e', e'', the quantities they represent.

196. If we have the three unknown quantities x, y, and z, and are furnished with an equal number of equations, distinguished by (1), (2) and (3); in order to determine these unknown quantities, we may combine, for example, the equation (1) with (2) and with (3), to eliminate x, and then exterminate y from the two results, which are obtained. But it must be observed, that by this *successive* elimination, the three proposed equations do not concur, in the same manner, to form the final equation; the equation (1) is employed twice, while (2) and (3) are employed only once; hence the result, to which we arrive, contains a factor foreign to the question (84). Bézout, in his *Théorie des Equations*, has made use of a method, which is not subject to this inconvenience, and by which he proves, that *the degree of the final equation, resulting from the elimination among any number whatever of complete equations, containing an equal number of unknown quantities, and quantities of any degrees whatever, is equal to the product of the exponents, which denote the degree of these equations.* M. Poisson, has given a demonstration of the same proposition more direct and shorter than that of Bézout; but the preliminary information, which it requires, will not permit me to explain it here; it will be found in the *Supplement.* At present, I shall observe simply, that it is easy to verify this proposition in the case of the final equations presented in articles 194 and 195. If we suppose the proposed equations given in those articles to be complete, the unknown quantity y enters of the first degree into P and P', of the second degree into Q and Q', of the third into R and R'; hence it follows, that e will be of the first degree, e' of the second, and e'' of the third, and that the terms of the highest degree found in the products indicated in the final equation given in art. 194, will have 4, or 2 . 2, for an exponent, and those of the final equation, art. 195, will have 9 or 3 . 3.

Of Commensurable Roots, and the equal Roots of Numerical Equations.

197. Having made known the most important properties of algebraic equations, and explained the method of eliminating the unknown quantities, when several occur, I shall proceed to the numerical resolution of equations with only one unknown quantity, that is, to the finding of their roots, when their coefficients are expressed by numbers.*

I shall begin by showing, that *when the proposed equation has only whole numbers for its coefficients, and that of its first term is unity, its real roots cannot be expressed by fractions, and consequently can be only whole numbers, or numbers, that are incommensurable.*

In order to prove this, let there be the equation

$$x^n + Px^{n-1} + Qx^{n-2} \ldots\ldots + Tx + U = 0,$$

in which we substitute for x an irreducible fraction $\frac{a}{b}$; the equation then becomes

$$\frac{a^n}{b^n} + P\frac{a^{n-1}}{b^{n-1}} + Q\frac{a^{n-2}}{b^{n-2}} \ldots\ldots + T\frac{a}{b} + U = 0;$$

reducing all the terms to the same denominator, we have

$$a^n + Pa^{n-1}b + Qa^{n-2}b^2 \ldots\ldots + Tab^{n-1} + Ub^n = 0,$$

which is equivalent to

$$a^n + b(Pa^{n-1} + Qa^{n-2}b \ldots\ldots + Tab^{n-2} + Ub^{n-1}) = 0.$$

The first member of this last equation consists of two entire parts, one of which is divisible by b, and the other is not (98), since it is supposed, that the fraction $\frac{a}{b}$ is reduced to its most simple form, or that a and b have no common divisor; one of these parts cannot therefore destroy the other.

198. After what has been said, we shall perceive the utility of making the fractions of an equation to disappear, or of rendering its coefficients entire numbers, in such a manner, however,

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* There is no general solution for degrees higher than the fourth; properly speaking, it is only that for the second degree, which can be regarded as complete. The expressions for the roots of equations of the third and fourth degree are very complicated, subject to exceptions, and less convenient in practice than those, which I am about to give; I shall resume the subject in the *Supplement*.

that the first term may have only unity for its coefficient. This is done *by making the unknown quantity proposed, equal to a new unknown quantity divided by the product of all the denominators of the equation*, then reducing all the terms to the same denominator, by the method given in art. 52.

Let there be, for example, the equation

$$x^3 + \frac{a x^2}{m} + \frac{b x}{n} + \frac{c}{p} = 0;$$

we take $x = \frac{y}{m n p}$, and introducing this expression for x into the proposed equation, we obtain

$$\frac{y^3}{m^3 n^3 p^3} + \frac{a y^2}{m^3 n^2 p^2} + \frac{b y}{m n^2 p} + \frac{c}{p} = 0;$$

as the divisor of the first term contains all the factors found in the other divisors, we may multiply by this divisor and thus reduce each term to its most simple expression; we find then

$$y^3 + a n p y^2 + b m^2 n p^2 y + c m^3 n^3 p^2 = 0.$$

If the denominators, m, n, p, have common divisors, it is only necessary to divide y by the least number, which can be divided at the same time by all the denominators. As these methods of simplifying expressions will be readily perceived, I shall not stop to explain them; I shall observe only, that if all the denominators were equal to m, it would be sufficient to make $x = \frac{y}{m}$.

The proposed equation, which would be in this case,

$$x^3 + \frac{a x^2}{m} + \frac{b x}{m} + \frac{c}{m} = 0,$$

then becomes

$$\frac{y^3}{m^3} + \frac{a y^2}{m^3} + \frac{b y}{m^2} + \frac{c}{m} = 0,$$

and we have

$$y^3 + a y^2 + b m y + m^2 c = 0.$$

It is evident, that the above operation amounts to multiplying all the roots of the proposed equations by the number m, since $x = \frac{y}{m}$ gives $y = m x$.

199. Now since, if a be the root of the equation

$$x^n + P x^{n-1} + Q x^{n-2} \dots + T x + U = 0,$$

we have

$$U = -a^n - P a^{n-1} - Q a^{n-2} \dots\dots - T a \ (179),$$

it follows, that a is necessarily one of the divisors of the entire number U, and consequently, when this number has but few divisors, we have only to substitute them successively in the place of x, in the proposed equation, in order to determine, whether or not this equation has any root among whole numbers.

If we have, for example, the equation

$$x^3 - 6x^2 + 27x - 38 = 0,$$

as the numbers

$$1, \quad 2, \quad 19, \quad 38,$$

are the only divisors of the number 38, we make trial of these, both in their positive and negative state; and we find, that the whole number $+ 2$ only satisfies the proposed equation, or that $x = 2$. We then divide the proposed equation by $x - 2$; putting the quotient equal to zero, we form the equation

$$x^2 - 4x + 19 = 0,$$

the roots of which are imaginary; and resolving this, we find that the proposed equation has three roots,

$$x = 2, \quad x = 2 + \sqrt{-15}, \quad x = 2 - \sqrt{-15}.$$

200. The method just explained, for finding the entire number, which satisfies an equation, becomes impracticable, when the last term of this equation has a great number of divisors; but the equation,

$$U = - a^n - P a^{n-1} - Q a^{n-2} \ldots . - T a,$$

furnishes new conditions, by means of which the operation may be very much abridged. In order to make the process more plain, I shall take, as an example, the equation

$$x^4 + P x^3 + Q x^2 + R x + S = 0.$$

The root being constantly represented by a, we have

$$a^4 + P a^3 + Q a^2 + R a + S = 0,$$
$$S = - R a - Q a^2 - P a^3 - a^4,$$

from which we obtain

$$\frac{S}{a} = - R - Q a - P a^2 - a^3.$$

It is evident from this last equation, that $\frac{S}{a}$ must be a whole number.

Bringing R into the first member, we have

$$\frac{S}{a} + R = - Q a - P a^2 - a^3;$$

abridging the expression by making $\frac{S}{a} + R = R'$, and dividing the two members of the equation

$$R' = -Qa - Pa^2 - a^3$$

by a, we have

$$\frac{R'}{a} = -Q - Pa - a^2,$$

whence we conclude, that $\frac{R'}{a}$ must also be a whole number.

Transposing Q, and making $\frac{R'}{a} + Q = Q'$, then dividing the two members by a, we obtain

$$\frac{Q'}{a} = -P - a,$$

whence we infer, that $\frac{Q'}{a}$ must be a whole number.

Lastly, bringing P into the first member, making $\frac{Q'}{a} + P = P'$, and dividing by a, we have

$$\frac{P'}{a} = -1.$$

Putting together the above mentioned conditions, we shall perceive that the number a will be the root of the proposed equation, if it satisfy the equations

$$\frac{S}{a} + R = R',$$
$$\frac{R'}{a} + Q = Q',$$
$$\frac{Q'}{a} + P = P',$$
$$\frac{P'}{a} + 1 = 0,$$

in such a manner, as to make R', Q', and P' whole numbers.

Hence it follows, that in order to determine, whether one of the divisors a of the last term S can be a root of the proposed equation, we must,

1st. *Divide the last term by the divisor* a, *and add to the quotient the coefficient of the term involving* x ;

2d. *Divide this sum by the divisor* a, *and add to the quotient the coefficient of the term involving* x^2 ;

3d. *Divide this sum by the divisor* a, *and add to the quotient the coefficient of the term involving* x^3 ;

4th. *Divide this sum by the divisor* a, *and add to the quotient unity, or the coefficient of the term involving* x^4; *the result will become equal to zero, if* a *is, in fact, the root.*

The rules given above are applicable, whatever be the degree of the equation; it must be observed, however, that the result will not become equal to zero, until we arrive at the first term of the proposed equation.*

201. In applying these rules to a numerical example, we may conduct the operation in such a manner as to introduce the several trials with all the divisors of the last term, at the same time.

For the equation

$$x^4 - 9x^3 + 23x^2 - 20x + 15 = 0,$$

the operation is, as follows;

$$\begin{array}{rrrrrrrr}
+15, & +5, & +3, & +1, & -1, & -3, & -5, & -15, \\
+1, & +3, & +5, & +15, & -15, & -5, & -3, & -1, \\
-19, & -17, & -15, & -5, & -35, & -25, & -23, & -21, \\
 & & -5, & -5, & +35, & & & \\
 & & +18, & +18, & +58, & & & \\
 & & +6, & +18, & -58, & & & \\
 & & -3, & +9, & -67, & & & \\
 & & -1, & +9, & +67, & & & \\
 & & 0. & & & & &
\end{array}$$

All the divisors of the last term 15 are arranged, in the order of magnitude, both with the sign + and —, and placed in the same line; this is the line occupied by the divisors a.

The second line contains the quotients arising from the number 15, divided successively by all its divisors; this is the line for the quantities $\frac{S}{a}$.

The third line is formed by adding to the numbers found in the

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* It would not be difficult to prove by means of the formula for the quotients given in art. 180, that the quantities $\frac{S}{a}, \frac{R'}{a}, \frac{Q'}{a}$, taken with the contrary sign, and with the order inverted, are the coefficients of the quotient arising from the polynomial

$$x^4 + Px^3 + Qx^2 + Rx + S$$

divided by $x - a$, and which is, consequently,

$$x^3 - \frac{Q'}{a}x^2 - \frac{R'}{a}x - \frac{S}{a}.$$

preceding the coefficient -20, by which x is multiplied; this is the line for the quantities $R' = \frac{S}{a} + R$.

The fourth line contains the quotients of the several numbers in the preceding, divided by the corresponding divisors; this is the line for the quantities $\frac{R}{a}$. In forming this line, we neglect all the numbers, which are not entire.

The fifth line results from the numbers, written in the preceding, added to the number 23, by which x^2 is multiplied; this line contains the quantities Q'.

The sixth line contains the quotients arising from the numbers in the preceding, divided by the corresponding divisors; it comprehends the quantities $\frac{Q'}{a}$.

The seventh line comprehends the several sums of the numbers in the preceding, added to the coefficient -9, by which x^3 is multiplied; in this line are found the quantities $\frac{Q'}{a} + P$.

Lastly, the eighth line is formed, by dividing the several numbers in the preceding by the corresponding divisors; it is the line for $\frac{P}{a}$. As we find -1 only in the column, at the head of which $+3$ stands, we conclude, that the proposed equation has only one commensurable root, namely, $+3$; it is, therefore, divisible by $x - 3$.*

The divisors $+1$ and -1 may be omitted in the table, as it is easier to make trial of them, by substituting them immediately in the proposed equation.

202. Again, let there be, for example,

$$x^3 - 7x^2 + 36 = 0.$$

Having ascertained, that the numbers $+1$ and -1 do not satisfy this equation, we form the table subjoined, according to the preceding rules, observing that, as the term involving x is wanting in this equation, x must be regarded as having 0 for a coefficient; we must, therefore, suppress the third line, and deduce the fourth immediately from the second.

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* Forming the quotient according to the preceding note, we find

$$x^3 - 6x^2 + 5x - 5.$$

$$\begin{array}{l}
+36, +18, +12, +9, +6, +4, +3, +2, -2, -3, -4, -6, -9, -12, -18, -36 \\
+1, +2, +3, +4, +6, +9, +12, +18, -18, -12, -9, -6, -4, -3, -2, -1
\end{array}$$

$$\begin{array}{cccccc}
+1, & +4, & +9, & +9, & +4, & +1, \\
-6, & -3, & +2, & +2, & -3, & -6, \\
-1, & -1, & & -1, & +1, & +1, \\
0, & 0, & & 0. & &
\end{array}$$

We find in this example three numbers, which fulfil all the conditions, namely, $+6$, $+3$, and -2. Thus we obtain, at the same time, the three roots, which the proposed equation admits of; we conclude then, that it is the product of three simple factors, $x - 6$, $x - 3$, and $x + 2$.

203. It may be observed, that there are literal equations, which may by transformed, at once, into numerical ones.

If we have, for example,

$$y^3 + 2py^2 - 33p^2y + 14p^3 = 0,$$

making $y = px$, we obtain

$$p^3x^3 + 2p^3x^2 - 33p^3x + 14p^3 = 0,$$

a result, which is divisible by p^3, and may be reduced to

$$x^3 + 2x^2 - 33x + 14 = 0.$$

As the commensurable divisor of this last equation is $x + 7$, which gives $x = -7$, we have

$$y = -7p.$$

The equation involving y is among those which are called *homogeneous equations*, because taken independently of the numerical coefficients, the several terms contain the same number of factors.*

204. When we have determined one of the roots of an equation, we may take for an unknown quantity the difference between this root and any one of the others; by this means we arrive at an equation of a degree inferior to that of the equation proposed, and which presents several remarkable properties.

Let there be the general equation

$$x^m + Px^{m-1} + Qx^{m-2} + Rx^{m-3} \ldots\ldots + Tx + U = 0,$$

and let a, b, c, d, &c. be its roots; substituting $a + y$ in the place of x, and developing the powers, we have

* For a more full account of the *commensurable divisors* of equations, the reader is referred to the third part of the *Elémens d' Algèbre* of Clairaut. This geometer has treated of literal as well as numerical equations.

$$\left.\begin{array}{l}a^m \quad + ma^{m-1}y \qquad + \quad \frac{m(m-1)}{2} \quad a^{m-2}y^2 + \ldots + y^m \\ + Pa^{m-1} + (m-1)Pa^{m-2}y + \frac{(m-1)(m-2)}{2}Pa^{m-3}y^2 + \ldots \\ + Qa^{m-2} + (m-2)Qa^{m-3}y + \frac{(m-2)(m-3)}{2}Qa^{m-4}y^2 + \ldots \\ + Ra^{m-3} + (m-3)Ra^{m-4}y + \frac{(m-3)(m-4)}{2}Ra^{m-5}y^2 + \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + Ta \quad + Ty \\ + U\end{array}\right\} = 0.$$

The first column of this result, being similar to the proposed equation, vanishes of itself, since a is one of the roots of this equation; we may, therefore, suppress this column, and divide all the remaining terms by y; the equation then becomes

$$\left.\begin{array}{l}\qquad m\,a^{m-1} + \quad \frac{m\,(m-1)}{2} \quad a^{m-2}\,y + \ldots + y^{m-1} \\ + (m-1)\,P\,a^{m-2} + \frac{(m-1)\,(m-2)}{2}\,P\,a^{m-3}\,y + \ldots \\ + (m-2)\,Q\,a^{m-3} + \frac{(m-2)\,(m-3)}{2}\,Q\,a^{m-4}\,y + \ldots \\ + (m-3)\,R\,a^{m-4} + \frac{(m-3)\,(m-4)}{2}\,R\,a^{m-5}\,y + \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + T\end{array}\right\} = 0.$$

This equation has evidently for its $m-1$ roots

$$y = b - a,\ y = c - a,\ y = d - a \ldots\ldots \&c.$$

I shall represent it by

$$A + \frac{B}{2}y + \frac{C}{2.3}y^2 \ldots\ldots\ldots + y^{m-1} = 0 \ldots\ldots\ldots (d),$$

abridging the expressions, by making

$$m\,a^{m-1} + (m-1)\,P\,a^{m-2} + (m-2)\,Q\,a^{m-3} \ldots\ldots + T = A,$$
$$m\,(m-1)\,a^{m-2} + (m-1)\,(m-2)\,P\,a^{m-3} \ldots\ldots \qquad = B,$$

&c.,

and I shall designate by V the expression

$$a^m + P\,a^{m-1} + Q\,a^{m-2} \ldots\ldots\ldots + T\,a + U.$$

205. If the proposed equation has two equal roots; if we have, for example, $a = b$, one of the values of y, namely, $b - a$, becomes nothing; the equation (d) will therefore be verified, by supposing $y = 0$; but upon this supposition all the terms vanish, except the known term A; this last must, therefore, be nothing of itself; the value of a must, therefore, satisfy, at the same time, the two equations

$$V = 0 \quad \text{and} \quad A = 0.$$

When the proposed equation has three roots equal to a, namely, $a = b = c$, two of the roots of the equation (d) become nothing, at the same time, namely, $b - a$ and $c - a$. In this case the equation (d) will be divisible twice successively by $y - 0$ (179) or y; but this can happen, only when the coefficients A and B are nothing; the value of a must then satisfy, at the same time, the three equations

$$V = 0, \quad A = 0, \quad B = 0.$$

Pursuing the same reasoning, we shall perceive, that when the proposed equation has four equal roots, the equation (d) will have three roots equal to zero, or will be divisible three times successively by y; the coefficients, A, B, and C, must then be nothing, at the same time, and consequently the value of a must satisfy at once the four equations,

$$V = 0, \quad A = 0, \quad B = 0, \quad C = 0.$$

By means of what has been said, we shall not only be able to ascertain, whether a given root is found several times among the roots of the proposed equation, but may deduce a method of determining, whether this equation has roots repeated, of which we are ignorant.

For this purpose, it may be observed, that when we have $A = 0$, or

$$m a^{m-1} + (m-1) P a^{m-2} + (m-2) Q a^{m-3} \ldots + T = 0,$$

we may consider a as the root of the equation

$$m x^{m-1} + (m-1) P x^{m-2} + (m-2) Q x^{m-3} \ldots + T = 0,$$

x representing, in this case, any unknown quantity whatever; and since a is also the root of the equation $V = 0$, or

$$x^m + P x^{m-1} + \&c. = 0,$$

it follows, (189) that $x - a$ is a factor common to the two above mentioned equations.

Changing in the same manner a into x in the quantities, B, C, &c. the binomial $x - a$ becomes likewise a factor of the two new equations, $B = 0$, $C = 0$, &c. if the root a reduces to nothing the original quantities, B, C, &c.

What has been said with respect to the root a, may be applied to every other root, which is several times repeated; thus, by seeking, according to the method given for finding the greatest common divisor, the factors common to the equations,

$$V = 0, \quad A = 0, \quad B = 0, \quad C = 0, \&c.,$$

we shall be furnished with the equal roots of the proposed equation, in the following order;

The factors common to the first two equations only, are twice factors in the equation proposed; that is, if we find for a common divisor of $V = 0$ and $A = 0$, an expression of the form $(x - \alpha)(x - \beta)$, for example, the unknown quantity x will have two values equal to α, and two equal to β, or the proposed equation will have these four factors,

$$(x - \alpha), \quad (x - \alpha), \quad (x - \beta), \quad (x - \beta).$$

The factors common, at the same time, to the first three of the above mentioned equations form triple factors in the proposed equation; that is, if the former are presented under the form $(x - \alpha)(x - \beta)$, the latter will take the form, $(x - \alpha)^3 (a - \beta)^3$. This reasoning may easily be extended to any length we please.

206. It may be remarked, that the equation $A = 0$, which, by by changing a into x, becomes

$$m x^{m-1} + (m-1) P x^{m-2} + (m-2) Q x^{m-3} \ldots + T = 0,$$

is deduced immediately from the equation $V = 0$, or from the proposed equation,

$$x^m + P x^{m-1} + Q x^{m-2} \ldots + T x + U = 0,$$

by multiplying each term of this last by the exponent of the power of x, which it contains, and then diminishing this exponent by unity. We may remark here, that the term U, which is equivalent to $U \times x^0$, is reduced to nothing in this operation, where it is multiplied by 0. The equation $B = 0$ is obtained from $A = 0$, in the same manner as $A = 0$ is deduced from $V = 0$; $C = 0$ is obtained from $B = 0$, in the same manner as this from $A = 0$, and so on.*

207. To illustrate what has been said, by an example, I shall take the equation

$$x^5 - 13 x^4 + 67 x^3 - 171 x^2 + 216 x - 108 = 0;$$

the equation $A = 0$ becomes, in this case,

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* It is shown, though very imperfectly, in most elementary treatises, that the divisor common to the two equations $V = 0$ and $A = 0$, contains equal factors raised to a power less by unity than that of the equation proposed; this may be readily inferred from what precedes; but for a demonstration of this proposition we refer the reader to the *Supplement*, where it is proved in a manner, which appears to me to be simple and new.

$$5x^4 - 52x^3 + 201x^2 - 342x + 216 = 0;$$

the divisor common to this and the proposed equation is

$$x^3 - 8x^2 + 21x - 18.$$

As this divisor is of the third degree, it must itself contain several factors; we must therefore seek, whether it does not contain some that are common to the equation $B = 0$, which is here

$$20x^3 - 156x^2 + 402x - 342 = 0.$$

We find, in fact, for a result $x - 3$; the proposed equation then has three roots equal to 3, or admits of $(x - 3)^3$ among the number of its factors. Dividing the first common divisor by $x - 3$, as many times as possible, that is, in this case twice, we obtain $x - 2$. As this divisor is common only to the proposed equation, and to the equation $A = 0$, it can enter only twice into the proposed equation. It is evident then, that this equation is equivalent to

$$(x - 3)^3 (x - 2)^2 = 0.$$

208. As the equation (d) gives the difference between b and the several other roots, when b is substituted for a, the difference between c and the others, when c is substituted for a, &c. and undergoes no change in its form by these several substitutions, retaining the coefficients belonging to the equation proposed, it may be converted into a general equation, which shall give all the differences between the several roots combined two and two. For this purpose, it is only necessary to eliminate a by means of the equation

$$a^m + Pa^{m-1} + Qa^{m-2} \ldots\ldots + Ta + U = 0;$$

for the result being expressed simply by the coefficients, and exhibiting the root under consideration in no form whatever, answers alike to all the roots.

It is evident, that the final equation must be raised to the degree $m(m - 1)$; for its roots

$$\begin{array}{llll} a - b, & a - c, & a - d, & \&c. \\ b - a, & b - c, & b - d, & \&c. \\ c - a, & c - b, & c - d, & \&c. \end{array}$$

are equal in number to the number of arrangements, which the m letters, a, b, c, &c. admit of when taken two and two. Moreover, since the quantities

$a - b$ and $b - a$, $a - c$ and $c - a$, $b - c$ and $c - b$, &c.

differ only in the sign, the roots of the equation are equal, when taken two and two, independently of the signs; so that if we have

$y = \alpha$, we shall have, at the same time, $y = -\alpha$. Hence it follows, that this equation must be made up of terms involving only even powers of the unknown quantity; for its first member must be the product of a certain number of factors of the second degree of the form

$$y^2 - \alpha^2 = (y - \alpha)(y + \alpha) \quad (184);$$

it will, therefore, itself be exhibited under the form

$$y^{2n} + p y^{2n-2} + q y^{2n-4} \ldots\ldots + t y^2 + u = 0.$$

If we put $y^2 = z$, this becomes

$$z^n + p z^{n-1} + q z^{n-2} \ldots\ldots + t z + u = 0;$$

and as the unknown quantity z is the square of y, its values will be the squares of the differences between the roots of the proposed equation.

It may be observed that as the differences between the real roots of the proposed equation are necessarily real, their squares will be positive, and consequently the equation in z will have only positive roots, if the proposed equation admits of those only, which are real.

Let there be, for example, the equation

$$x^3 - 7x + 7 = 0;$$

putting $x = a + y$, we have

$$\left.\begin{array}{l} a^3 + 3a^2 y + 3 a y^2 + y^3 \\ -7a - 7y \\ +7 \end{array}\right\} = 0.$$

Suppressing the terms $a^3 - 7a + 7$, which, from their identity with the proposed equation, become nothing when united, and dividing the remainder by y, we have

$$3a^2 + 3ay + y^2 - 7 = 0;$$

eliminating a by means of this equation and the equation

$$a^3 - 7a + 7 = 0,$$

we have

$$y^6 - 42 y^4 + 441 y^2 - 49 = 0;$$

putting $z = y^2$, this becomes

$$z^3 - 42 z^2 + 441 z - 49 = 0.$$

209. The substitution of $a + y$ in the place of x in the equation

$$x^m + P x^{m-1} + Q x^{m-2} \ldots\ldots + U = 0 \quad (204),$$

is sometimes resorted to also in order to make one of the terms of this equation to disappear. We then arrange the result with reference to the powers of y, which takes the place of the unknown quantity x, and consider a as a second unknown quan-

tity, which is determined by putting equal to zero the coefficient of the term we wish to cancel; in this way we obtain

$$\left.\begin{array}{l} y^m + m a y^{m-1} + \dfrac{m(m-1)}{1\,.\,2} a^2 y^{m-2} \ldots\ldots + a^m \\ \quad + \; P y^{m-1} + (m-1) P a y^{m-2} \ldots\ldots + P a^{m-1} \\ \qquad\qquad + \qquad\qquad Q y^{m-2} \ldots\ldots + Q a^{m-2} \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \qquad\qquad\qquad\qquad\qquad\qquad + U \end{array}\right\} = 0.$$

If the term we would suppress be the second, or that which involves y^{m-1}, we make $m a + P = 0$, from which we deduce $a = -\frac{P}{m}$. Substituting this value in the result, there remain only the terms involving

$$y^m, \quad y^{m-2}, \quad y^{m-3}, \quad \&c.$$

Hence it follows, that *we make the second term of an equation to disappear, by substituting for the unknown quantity in this equation a new unknown quantity, united with the coefficient of the second term taken with the sign contrary to that originally belonging to it, and divided by the exponent of the first term.*

Let there be, for example, the equation

$$x^3 + 6x^2 - 3x + 4 = 0;$$

we have by the rule

$$x = y - \tfrac{6}{3} = y - 2;$$

substituting this value, the equation becomes

$$\left.\begin{array}{r} y^3 - 6y^2 + 12y - 8 \\ + 6y^2 - 24y + 24 \\ - 3y + 6 \\ + 4 \end{array}\right\} = 0,$$

which may be reduced to

$$y^3 - 15y + 26 = 0,$$

in which the term involving y^2 does not appear. We may cause the third term, or that involving y^{m-2}, to disappear by putting equal to zero the sum of the quantities, by which it is multiplied, that is, by forming the equation

$$\frac{m(m-1)}{1\,.\,2} a^2 + (m-1) P a + Q = 0.$$

Pursuing this method, we shall readily perceive, that the fourth term will be made to vanish by means of an equation of the third degree, and so on to the last, which can be made to disappear only by means of the equation

$$a^m + P a^{m-1} + Q a^{m-2} \ldots\ldots + U = 0,$$

perfectly similar to the equation proposed.

It is not difficult to discover the reason of this similarity. By making the last term of the equation in y equal to zero, we suppose, that one of the values of this unknown quantity is zero; and if we admit this supposition with respect to the equation $x = y + a$, it follows that $x = a$; that is, the quantity a, in this case, is necessarily one of the values of x.

210. We have sometimes occasion to resolve equations into factors of the second and higher degrees. I cannot here explain in detail the several processes, which may be employed for this purpose; one example only will be given.

Let there be the equation

$$x^4 - 24x^3 + 12x^2 - 11x + 7 = 0,$$

in which it is required to determine the factors of the third degree; I shall represent one of these factors by

$$x^3 + px^2 + qx + r,$$

the coefficients, p, q, and r, being indeterminate. They must be such, that the first member of the proposed equation will be exactly divisible by the factor

$$x^3 + px^2 + qx + r,$$

independently of any particular value of x; but in making an actual division, we meet with a remainder

$$\begin{aligned} &-(p^3 - 2pq - 24p + r - 12)x^2 \\ &-(p^2 q - pr - q^2 - 24q + 11)x \\ &-(p^2 r - qr - 24r - 7), \end{aligned}$$

an expression, which must be reduced to nothing, independently of x, when we substitute for the letters, p, q, and r, the values that answer to the conditions of the question. We have then

$$\begin{aligned} p^3 - 2pq - 24p + r - 12 &= 0, \\ p^2 q - pr - q^2 - 24q + 11 &= 0, \\ p^2 r - qr - 24r - 7 &= 0. \end{aligned}$$

These three equations furnish us with the means of determining the unknown quantities, p, q, and r; and it is to a resolution of these, that the proposed question is reduced.

Of the Resolution of Numerical Equations by Approximation.

211. Having completed the investigation of commensurable divisors, we must have recourse to the methods of finding roots by approximation, which depend on the following principle;

When we arrive at two quantities which, substituted in the place of the unknown quantity in an equation, lead to two results with contrary signs, we may infer, that one of the roots of the proposed equation lies between these two quantities, and is consequently real.

Let there be, for example, the equation

$$x^3 - 13x^2 + 7x - 1 = 0;$$

if we substitute, successively, 2 and 20 in the place of x, in the first member, instead of being reduced to zero, this member becomes, in the former case, equal to -31, and in the latter, to $+2939$; we may therefore conclude, that this equation has a real root between 2 and 20, that is, greater than two and less than 20.

As there will be frequent occasion to express this relation, I shall employ the signs $>$ and $<$, which algebraists have adopted to denote the inequality of two magnitudes, placing the greater of two quantities opposite the opening of the lines, and the less against the point of meeting. Thus I shall write

$x > 2$, to denote, that x is greater than 2,

$x < 20$, to denote, that x is less than 20.

Now in order to prove what has been laid down above, we may reason in the following manner. Bringing together the positive terms of the proposed equation, and also those which are negative, we have

$$x^3 + 7x - (13x^2 + 1),$$

a quantity, which will be negative, if we suppose $x = 2$, because, upon this supposition,

$$x^3 + 7x < 13x^2 + 1,$$

and which becomes positive, when we make $x = 20$, because, in this case,

$$x^3 + 7x > 13x^2 + 1.$$

Moreover, it is evident, that the quantities

$$x^3 + 7x \quad \text{and} \quad 13x^2 + 1,$$

each increase, as greater and greater values are assigned to x, and that, by taking values, which approach each other very nearly, we may make the increments of the proposed quantities as small as we please. But since the first of the above quantities, which was originally less than the second, becomes greater, it is evident, that it increases more rapidly than the other, in consequence of which its deficiency is made up, and it comes at length

to exceed the other; there must, therefore, be a point at which the two magnitudes are equal.

The value of x, whatever it be, which renders
$$x^3 + 7x = 13x^2 + 1,$$
and such a value has been proved to exist, gives
$$x^3 + 7x - (13x^2 + 1) = 0,$$
or
$$x^3 - 13x^2 + 7x - 1 = 0,$$
and must necessarily, therefore, be the root of the equation proposed.

What has been shown with respect to the particular equation
$$x^3 - 13x^2 + 7x - 1 = 0,$$
may be affirmed of any equation whatever, the positive terms of which I shall designate by P, and the negative by N. Let a be the value of x, which leads to a negative result, and b that which leads to a positive one; these consequences can take place only upon the supposition, that by substituting the first value, we have $P < N$, and by substituting the second, $P > N$; P, therefore, from being less, having become greater than N, we conclude as above, that there exists a value of x between a and b, which gives $P = N$.*

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* The above reasoning, though it may be regarded as sufficiently evident, when considered in a general view, has been developed by M. Encontre in a manner, that will be found to be useful to those, who may wish to see the proofs given more in detail.

1. It is evident, that the increments of the polynomials P and N may be made as small as we please. Let
$$P = \alpha x^m + 6 x^n \ldots . + \delta x^q,$$
m being the highest exponent of x; if we put $a + y$ in the place of x, this polynomial takes the form
$$A + By + Cy^2 \ldots\ldots + Ty^m,$$
the coefficients, A, B, C, ... T, being finite in number and having a finite value; the first term A will be the value the polynomial P assumes, when $x = a$; the remainder,
$$By + Cy^2 \ldots . + Ty^m = y\,(B + Cy \ldots . Ty^{m-1}),$$
will be the quantity, by which the same polynomial is increased when we augment by y the value $x = a$. This being admitted, if S designate the greatest of the coefficients, B, C, ... T, we have
$$B + Cy \ldots . + Ty^{m-1} < S\,(1 + y \ldots . + y^{m-1});$$
now
$$1 + y \ldots . + y^{m-1} = \frac{1 - y^m}{1 - y} \quad (158);$$

The statement here given seems to require, that the values assigned to x should be both positive or both negative, for if they have different signs, that which is negative produces a change in the signs of those terms of the proposed equation, which contain odd powers of the unknown quantity, and, consequently, the expressions P and N are not formed in the same manner, when we substitute one value, as when we substitute the other. This difficulty vanishes, if we make $x = 0$; in this case, the proposed equation reduces itself to its last term, which has necessarily a sign contrary to that of the result arising from the substitution of one or the other of the above mentioned values. Let there be, for example, the equation

$$x^4 - 2x^3 - 3x^2 - 15x - 3 = 0,$$

the first member of which, when we put

$$x = -1 \quad \text{and} \quad x = 2,$$

becomes $+ 12$ and $- 45$. If we suppose $x = 0$, this member is reduced to $- 3$; substituting, therefore,

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therefore,

$$y\,(B + Cy \ldots . + Ty^{m-1}) < Sy\,\frac{(1 - y^m)}{1 - y},$$

and, consequently, the quantity by which the polynomial P is increased, will be less than any given quantity m, if we make $\frac{Sy\,(1 - y^m)}{1 - y}$ less than this last quantity; this is effected by making $\frac{Sy}{1 - y} = m$, because, in this case, $y \frac{m}{S + m}$ being < 1, the quantity $\frac{Sy\,(1 - y^m)}{1 - y}$, equal to $\frac{Sy}{1 - y} - \frac{Sy^{m+1}}{1 - y}$, will necessarily be less than the quantity m, which is indefinitely small.

2. If we designate by h the increment of the polynomial P, and by k that of the polynomial N, the change, which will be produced in the value of their difference, will be $h - k$, and may be rendered smaller than a given quantity, by making smaller than this same quantity the increment, which is the greater of the two; we may, therefore, in the interval between $x = a$ and $x = b$, take values, which shall make the difference of the polynomials P and N change by quantities as small as we please, and since this difference passes in this interval from positive to negative, it may be made to approach as near to zero as we choose. See *Annales de Mathématiques pures et appliquées*, published by M. Gergonne, vol. iv. p. 210.

$$x = 0 \quad \text{and} \quad x = -1,$$

we arrive at two results with contrary signs; but putting $-y$ in the place of x, the proposed equation is changed to

$$y^4 + 2y^3 - 3y^2 + 15y - 3 = 0,$$

and we have

$$P = y^4 + 2y^3 + 15y, \quad N = 3y^2 + 3,$$

whence

$$P < N, \text{ when } y = 0,$$
$$P > N, \text{ when } y = 1.$$

Reasoning as before, we may conclude, that the equation in y has a real root, found between 0 and $+1$; whence it follows, that the root of the equation in x lies between 0 and -1, and, consequently, between $+2$ and -1.

As every case the proposition enunciated can present, may be reduced to one or the other of those which have been examined, the truth of this proposition is sufficiently established.

212. Before proceeding further, I shall observe, that *whatever be the degree of an equation, and whatever its coefficients, we may always assign a number, which, substituted for the unknown quantity, will render the first term greater than the sum of all the others.* The truth of this proposition will be immediately apparent from what has been intimated of the rapidity, with which the several powers of a number greater than unity increase (126); since the highest of these powers exceeds those below it more and more in proportion to the increased magnitude of the number employed, so that there is no limit to the excess of the first above each of the others. Observe, moreover, the method by which we may find a number that fulfils the condition required by the enunciation.

It is evident, that the case most unfavourable to the supposition, is that, in which we make all the coefficients of the equation negative, and each equal to the greatest, that is, when instead of

$$x^m + Px^{m-1} + Qx^{m-2} \dots\dots + Tx + U = 0,$$

we take

$$x^m - Sx^{m-1} - Sx^{m-2} \dots\dots - Sx - S = 0,$$

S representing the greatest of the coefficients, $P, Q, \dots\, T, U$. Giving to the first member of this equation the form

$$x^m - S(x^{m-1} + x^{m-2} \dots\dots + 1),$$

we may observe, that

$$x^{m-1} + x^{m-2} \dots + 1 = \frac{x^m - 1}{x - 1} \quad (158);$$

the preceding expression then may be changed into

$$x^m - \frac{S(x^m - 1)}{x - 1}, \text{ or into } x^m - \frac{Sx^m}{x-1} + \frac{S}{x-1}.$$

If we substitute M for x, this becomes

$$M^m - \frac{SM^m}{M-1} + \frac{S}{M-1},$$

a quantity, which evidently becomes positive, if we make

$$M^m = \frac{S\,M^m}{M-1}.$$

Now if we divide each member of this equation by M^m, we have

$$1 = \frac{S}{M-1} \quad \text{or} \quad M = S + 1.$$

By substituting therefore for x the greatest of the coefficients found in the equation, augmented by unity, we render the first term greater than the sum of all the others.

A smaller number may be taken for M, if we wish simply to render the positive part of the equation greater than the negative; for to do this, it is only necessary to render the first term greater than the sum arising from all the others, when their coefficients are each equal, not to the greatest among all the coefficients, but to the greatest of those which are negative; we have, therefore, merely to take for M this coefficient augmented by unity.*

Hence it follows, that the positive roots of the proposed equation are necessarily comprehended within 0 and $S + 1$.

In the same way we may discover a limit to the negative roots; for this purpose we must substitute $-y$ for x, in the proposed equation, and render the first term positive, if it becomes negative (178). It is evident, that by a transformation of this kind, the positive values of y answer to the negative values of x, and the reverse. If R be the greatest negative coefficient after this change, $R + 1$ will form a limit to the positive values of y; consequently $-R - 1$ will form that of the negative values of x.

Lastly, if we would find for the smallest of the roots a limit approaching as near to zero as possible, we may arrive at it by

* In the *Résolution des équations numériques*, by Lagrange, there are formulas, which reduce this number to narrower limits, but what has been said above is sufficient to render the fundamental propositions for the resolution of numerical equations independent of the consideration of infinity.

substituting $\frac{1}{y}$ for x in the proposed equation, and preparing the equation in y, which is thus obtained, according to the directions given in art. 178. As the values of y are the reverse of those of x, the greatest of the first will correspond to the least of the second, and, reciprocally, the greatest of the second to the least of the first. If, therefore, $S' + 1$ represent the highest limit to the values of y, that is, if

$$y < S' + 1;$$

which gives

$$\frac{1}{x} < S' + 1,$$

we shall have, successively,

$$1 < (S' + 1)\,x, \; \frac{1}{S' + 1} < x.$$

Indeed, it is very evident, that we may, without altering the relative magnitude of two quantities separated by the sign $<$ or $>$, multiply or divide them by the same quantity, and that we may also add the same quantity to or subtract it from each side of the signs $<$ and $>$, which possess, in this respect, the same properties as the sign of equality.

213. It follows from what precedes, that *every equation of a degree denoted by an odd number has necessarily a real root affected with a sign contrary to that of its last term;* for if we take the number M such, that the sign of the quantity

$$M^m + PM^{m-1} + QM^{m-2} \ldots\ldots + TM \pm U,$$

depends solely on that of its first term M^m, the exponent m being an odd number, the term M^m will have the same sign as the number M (128). This being admitted, if the last term U has the sign $+$, and we make $x = -M$, we shall arrive at a result affected with a sign contrary to that, which the supposition of $x = 0$ would give; from which it is evident, that the proposed equation has a root between 0 and $-M$, that is, a negative root. If the last term U has the sign $-$, we make $x = +M$; the result will then have a sign contrary to that given by the supposition of $x = 0$, and in this case, the root will be found between 0 and $+M$, that is, it will be positive.

214. When the proposed equation is of a degree denoted by an even number, as the first term M^m remains positive, whatever sign we give to M, we are not, by the preceding observations, furnished with the means of proving the existence of a real root,

if the last term has the sign $+$, since, whether we make $x = 0$, or $x = \pm M$, we have always a positive result. But when this term is negative, we find, by making

$$x = + M, \quad x = 0, \quad x = - M,$$

three results, affected respectively with the signs $+$, $-$, and $+$, and, consequently, the proposed equation has, at least, two real roots in this case, the one positive, found between M and 0, the other negative, between 0 and $-M$; therefore, *every equation of an even degree, the last term of which is negative, has at least two real roots, the one positive and the other negative.*

215. I now proceed to the resolution of equations by approximation; and in order to render what is to be offered on this subject more clear, I shall begin with an example.

Let there be the equation

$$x^4 - 4x^3 - 3x + 27 = 0;$$

the greatest negative coefficient found in this equation being -4, it follows (212), that the greatest positive root will be less than 5. Substituting $-y$ for x, we have

$$y^4 + 4y^3 + 3y + 27 = 0;$$

and as all the terms of this result are positive, it appears, that y must be negative; whence it follows, that x is necessarily positive, and that the proposed equation can have no negative roots; its real roots are, therefore, found between 0 and $+5$.

The first method, which presents itself for reducing the limits, between which the roots are to be sought, is to suppose successively

$$x = 1, \quad x = 2, \quad x = 3, \quad x = 4;$$

and if two of these numbers, substituted in the proposed equation, lead to results with contrary signs, they will form new limits to the roots. Now if we make

$x = 1$, the first member of the equation becomes $+ 21$,
$x = 2$ . . . . $+ 5$,
$x = 3$ . . . . $- 9$,
$x = 4$ . . . . $+ 15$;

it is evident, therefore, that this equation has two real roots, the one found between 2 and 3, and the other between 3 and 4. To approximate the first still nearer, we take the number 2,5, which occupies the middle place between 2 and 3 (*Arith.* 129), the present limits of this root; making then $x = 2{,}5$, we arrive at the result

$$+ 39{,}0625 - 62{,}5 - 7{,}5 + 27 = - 3{,}9375;$$

as this result is negative, it is evident, that the root sought is between 2 and 2,5. The mean of these two numbers is 2,25; taking $x = 2{,}3$ we have the root sought within about one tenth of its value, and shall approximate the true root very fast by the following process, given by Newton.

We make $x = 2{,}3 + y$; it is evident, that the unknown quantity y amounts only to a very small fraction, the square and higher powers of which may be neglected; we have then

$$\begin{aligned} x^4 &= \quad (2{,}3)^4 + \ 4\,(2{,}3)^3 y \\ -4x^3 &= -4\,(2{,}3)^3 - 12\,(2{,}3)^2 y \\ -3x &= -3\,(2{,}3) \ - \ \ 3y; \end{aligned}$$

substituting these values, the proposed equation becomes

$$- 0{,}5839 - 17{,}812\,y = 0,$$

which gives

$$y = -\frac{0{,}5839}{17{,}812}.$$

Stopping at hundredths, we obtain for the result of the first operation

$$y = - 0{,}03 \text{ and } x = 2{,}3 - 0{,}03 = 2{,}27.$$

To obtain a new value of x, more exact than the preceding, we suppose $x = 2{,}27 + y'$; substituting this value in the proposed equation and neglecting all the powers of y' exceeding the first, we find

$$- 0{,}04595359 - 18{,}046468\,y' = 0,$$

whence

$$y' = -\frac{0{,}04595359}{18{,}046468} = - 0{,}0025,$$

and, consequently, $x = 2{,}2675$. We may, by pursuing this process, approximate, as nearly as we please, the true value of x.

If we seek the second root, contained between 3 and 4, by the same method, we find, stopping at the fourth decimal place,

$$x = 3{,}6797.$$

216. We may ascertain the exactness of the method above explained, by seeking the limit to the values of the terms, which are neglected.

If the proposed equation were

$$x^m + P\,x^{m-1} + Q\,x^{m-2} \ldots\ldots + T\,x + U = 0,$$

substituting $a + y$ for x, we should have for the result the first of the equations found in art. 204, because a being not the root of

the equation, but only an approximate value of x, cannot reduce to nothing the quantity

$$a^m + Pa^{m-1} + Qa^{m-2} \ldots\ldots\ldots + Ta + U.$$

Representing this last by V, we have, instead of the equation (d) above referred to, the following

$$V + \frac{A}{1}y + \frac{B}{1.2}y^2 + \frac{C}{1.2.3}y^3 \ldots\ldots + y^m = 0;$$

from which we obtain

$$Ay = -V - \frac{B}{1.2}y^2 - \frac{C}{1.2.3}y^3 \ldots\ldots\ldots - y^m,$$

$$y = -\frac{V}{A} - \frac{By^2}{1.2A} - \frac{Cy^3}{1.2.3A} \ldots\ldots\ldots - \frac{y^m}{A}.$$

Neglecting the powers of y exceeding the first, we have

$$y = -\frac{V}{A},$$

and this value differs from the real value of y by

$$-\frac{By^2}{1.2A} - \frac{Cy^3}{1.2.3A} \ldots\ldots - \frac{y^m}{A}.$$

If a differs from the true value of x only by a quantity less than $\frac{1}{p}a$, the above mentioned error becomes less than that, which would arise from putting $\frac{1}{p}a$ in the place of y, which would give

$$-\frac{B}{1.2A}\left(\frac{a}{p}\right)^2 - \frac{C}{1.2.3A}\left(\frac{a}{p}\right)^3 \ldots\ldots - \frac{1}{A}\left(\frac{a}{p}\right)^m.$$

Finding the value of this quantity, we shall be able to determine, whether it may be neglected when considered with reference to $\frac{V}{A}$, and if it be found too large, we must obtain for a a number, which approaches nearer to the true value of x.

To conclude, when we have gone through the calculation with several numbers, y, y', y'', &c. if the results thus obtained form a decreasing series, an approximation is certain.

217. The method we have employed above, is called the *Method by successive Substitutions*. Lagrange has considerably improved it.* He has remarked, that by substituting only entire

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* See *Résolution des Equations numériques.*

numbers, we may pass over several roots without perceiving them. In fact, if we have, for example, the equation

$$(x - \tfrac{1}{3})(x - \tfrac{1}{2})(x - 3)(x - 4) = 0,$$

by substituting for x the numbers, 0, 1, 2, 3, &c. we shall pass over the roots $\frac{1}{3}$ and $\frac{1}{2}$, without discovering that they exist; for we shall have

$$(0 - \tfrac{1}{3})(0 - \tfrac{1}{2})(0 - 3)(0 - 4) = + \tfrac{1}{3} \times \tfrac{1}{2} \times 3 \times 4,$$
$$(1 - \tfrac{1}{3})(1 - \tfrac{1}{2})(1 - 3)(1 - 4) = + \tfrac{2}{3} \times \tfrac{1}{2} \times 2 \times 3,$$

results affected by the same sign. It will be readily perceived, that this circumstance takes place in consequence of the fact, that the substitution of 1 for x changes at the same time the signs of both the factors, $x - \frac{1}{3}$, and $x - \frac{1}{2}$, which pass from the negative state, in which they are when 0 is put in the place of x, to the positive; but if we substitute for x a number between $\frac{1}{3}$ and $\frac{1}{2}$, the sign of the factor $x - \frac{1}{3}$ alone will be changed, and we shall obtain a negative result.

We shall necessarily meet with such a number, if we substitute, in the place of x, numbers, which differ from each other by a quantity less than the difference between the roots $\frac{1}{3}$ and $\frac{1}{2}$. If, for example, we substitute $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, &c. there will be two changes of the sign.

It may be objected to the above example, that when the fractional coefficients of an equation have been made to disappear, the equation can have for roots only either entire or irrational numbers, and not fractions; but it will be readily seen, that the irrational numbers, for which we have, in the example, substituted fractions for the purpose of simplifying the expressions, may differ from each other by a quantity less than unity.

In general, the results will have the same sign, whenever the substitutions produce a change in the sign of an even number of factors.* To obviate this inconvenience we must take the numbers to be substituted, such, that the difference between the smallest limit and the greatest, will be less than the least of the differences, which can exist between the roots of the proposed equation; by this means the numbers to be substituted will necessa-

* Equal roots cannot be discovered by this process, when their number is even; to find these we must employ the method given in art. 205.

rily fall between the successive roots, and will cause a change in the sign of one factor only. This process does not presuppose the smallest difference between the roots to be known, but requires only, that the limit, below which it cannot fall, be determined.

In order to obtain this limit, we form the equation involving the squares of the differences of the roots (208).

Let there be the equation

$$z^n + p z^{n-1} + q z^{n-2} \dots + tz + u = 0 \dots (D),$$

to obtain the smallest limit to the roots, we make (212) $z = \frac{1}{v}$; we have then the equation

$$\frac{1}{v^n} + p \frac{1}{v^{n-1}} + q \frac{1}{v^{n-2}} \dots + t \frac{1}{v} + u = 0,$$

or, reducing all the terms to the same denominator,

$$1 + pv + qv^2 \dots\dots + tv^{n-1} + uv^n = 0,$$

then disengaging v^n,

$$v^n + \frac{t}{u} v^{n-1} \dots + \frac{q}{u} v^2 + \frac{p}{u} v + \frac{1}{u} = 0;$$

and if $\frac{r}{u}$ represent the greatest negative coefficient found in this equation, we shall have

$$\frac{1}{\frac{r}{u} + 1} < z.$$

It is only necessary to consider here the positive limit, as this alone relates to the real roots of the proposed equation.

Knowing the limit

$$\frac{1}{\frac{r}{u} + 1} = \frac{u}{r + u},$$

less than the square of the smallest difference between the roots of the proposed equation, we may find its square root, or at least, take the rational number next below this root; this number, which I shall designate by k, will represent the difference which must exist between the several numbers to be substituted. We thus form the two series,

$$0, \ + k, \ + 2k, \ + 3k, \ \&c.$$
$$- k, \ - 2k, \ - 3k, \ \&c.$$

from which we are to take only the terms, comprehended between the limits to the smallest and the greatest positive roots,

and those to the smallest and the greatest negative roots of the proposed equation. Substituting these different numbers, we shall arrive at a series of results, which will show by the changes of the sign that take place, the several real roots, whether positive or negative.

218. Let there be, for example, the equation

$$x^3 - 7x + 7 = 0,$$

from which, in art. 208, was derived the equation

$$z^3 - 42z^2 + 441z - 49 = 0;$$

making $z = \frac{1}{v}$, and, after substituting this value, arranging the result with reference to v, we have

$$v^3 - 9v^2 + \tfrac{42}{49}v - \tfrac{1}{49} = 0,$$

from which we obtain

$$v < 10,\ z > \tfrac{1}{10};$$

we must, therefore, take $k =$ or $< \frac{1}{\sqrt{10}}$. This condition will be fulfilled, if we make $k = \frac{1}{4}$; but it is only necessary to suppose $k = \frac{1}{3}$; for by putting 9 in the place of v in the preceding equation, we obtain a positive result, which must become greater, when a greater value is assigned to v, since the terms v^3 and $9v^2$ already destroy each other, and $\frac{42}{49}v$ exceeds $\frac{1}{49}$.

The highest limit to the positive roots of the proposed equation

$$x^3 - 7x + 7 = 0,$$

is 8, and that to the negative roots -8; we must, therefore, substitute for x the numbers

$$0,\quad \tfrac{1}{3},\quad \tfrac{2}{3},\quad \tfrac{3}{3},\quad \tfrac{4}{3}, \ldots\ldots\quad \tfrac{24}{3},$$
$$-\tfrac{1}{3}, -\tfrac{2}{3}, -\tfrac{3}{3}, -\tfrac{4}{3}, \ldots\ldots -\tfrac{24}{3}.$$

We may avoid fractions by making $x = \frac{x'}{3}$; for in this case the differences between the several values of x' will be triple of those between the values of x, and, consequently, will exceed unity; we shall then have only to substitute, successively,

$$0,\quad 1,\quad 2,\quad 3, \ldots\ldots\quad 24,$$
$$-1, -2, -3, \ldots\ldots -24,$$

in the equation

$$x'^3 - 63x' + 189 = 0.$$

The signs of the results will be changed between $+4$ and $+5$, between $+5$ and $+6$, and between -9 and -10, so that we shall have for the positive values,

$$\left.\begin{array}{l} x' > 4 \text{ and } < 5 \\ x' > 5 \text{ and } < 6 \end{array}\right\} \text{ whence } \left\{\begin{array}{l} x > \frac{4}{3} \text{ and } < \frac{5}{3} \\ x > \frac{5}{3} \text{ and } < \frac{6}{3} \end{array}\right.$$

and the negative value of x' will be found between -9 and -10, that of x between $-\frac{9}{3}$ and $-\frac{10}{3}$.

Knowing now the several roots of the proposed equation within $\frac{1}{3}$, we may approach nearer to the true value by the method explained in art. 215.

219. The methods employed in the example given in art. 215, and in the preceding article, may be applied to an equation of any degree whatever, and will lead to values approaching the several real roots of this equation. It must be admitted, however, that the operation becomes very tedious, when the degree of the proposed equation is very elevated; but in most cases it will be unnecessary to resort to the equation (D), or rather its place may be supplied by methods, with which the study of the higher branches of analysis will make us acquainted.*

I shall observe, however, that by substituting successively the numbers, 0, 1, 2, 3, &c. in the place of x, we shall often be lead to suspect the existence of roots, that differ from each other by a quantity less than unity. In the example, upon which we have been employed, the results are

$$+7,\ +1,\ +1,\ +13,$$

which begin to increase after having decreased from $+7$ to $+1$. From this order being reversed it may be supposed, that between the numbers $+1$ and $+2$ there are two roots either equal or nearly equal. To verify this supposition, the unknown quantity should be multiplied. Making $x = \frac{y}{10}$, we find

$$y^3 - 700y + 7000 = 0,$$

an equation, which has two positive roots, one between 13 and 14, and the other between 16 and 17.

The number of trials necessary for discovering these roots is not great; for it is only between 10 and 20, that we are to search for y; and the values of this unknown quantity being deter-

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* A very elegant method, given by Lagrange for avoiding the use of the equation (D) may be found in the *Traité de la Résolution des Equations numériques.*

mined in whole numbers, we may find those of x within one tenth of unity.

220. When the coefficients in the equation proposed for resolution are very large, it will be found convenient to transform this equation into another, in which the coefficients shall be reduced to smaller numbers. If we have, for example,

$$x^4 - 80x^3 + 1998x^2 - 14937x + 5000 = 0,$$

we may make $x = 10z$; the equation then becomes

$$z^4 - 8z^3 + 19{,}98z^2 - 14{,}937z + 0{,}5 = 0.$$

If we take the entire numbers, which approach nearest to the coefficients in this result, we shall have

$$z^4 - 8z^3 + 20z^2 - 15z + 0{,}5 = 0.$$

It may be readily discovered, that z has two real values, one between 0 and 1, the other between 1 and 2, whence it follows, that those of the proposed equation are between 0 and 10, and 10 and 20.

I shall not here enter into the investigation of imaginary roots, as it depends on principles we cannot at present stop to illustrate; I shall pursue the subject in the *Supplement.*

221. Lagrange has given to the successive substitutions a form which has this advantage, that it shows immediately what approaches we make to the true root by each of the several operations, and which does not presuppose the value to be known within one tenth.

Let a represent the entire number immediately below the root sought; to obtain this root, it will be only necessary to augment a by a fraction; we have, therefore, $x = a + \frac{1}{y}$. The equation involving y, with which we are furnished by substituting this value in the proposed equation, will necessarily have one root greater than unity; taking b to represent the entire number immediately below this root, we have for the second approximation $x = a + \frac{1}{b}$. But b having the same relation to y, which a has to x, we may, in the equation involving y, make $y = b + \frac{1}{y'}$, and y' will necessarily be greater than unity; representing by b' the entire number immediately below the root of the equation in y', we have

$$y = b + \frac{1}{b'} = \frac{bb' + 1}{b'};$$

substituting this value in the expression for x, we have

$$x = a + \frac{b'}{b\,b' + 1},$$

for the third approximation to x. We may find a fourth by making $y' = b' + \frac{1}{y''}$; for if b'' designate the entire number immediately below y'', we shall have

$$y' = b' + \frac{1}{b''} = \frac{b'\,b'' + 1}{b''},$$

whence

$$y = b + \frac{b''}{b'\,b'' + 1} = \frac{b\,b'\,b'' + b'' + b}{b'\,b'' + 1},$$

$$x = a + \frac{b'\,b'' + 1}{b\,b'\,b'' + b'' + b},$$

and so on.

222. I shall apply this method to the equation

$$x^3 - 7x + 7 = 0.$$

We have already seen (218), that the smallest of the positive roots of this equation is found between $\frac{4}{3}$ and $\frac{5}{3}$, that is, between 1 and 2; we make, therefore, $x = 1 + \frac{1}{y}$; we shall then have

$$y^3 - 4y^3 + 3y + 1 = 0.$$

The limit to the positive roots of this last equation is 5, and by substituting, successively, 0, 1, 2, 3, 4, in the place of y, we immediately discover, that this equation has two roots greater than unity, one between 1 and 2; and the other between 2 and 3. Hence

$$x = 1 + \tfrac{1}{1}, \quad \text{and} \quad x = 1 + \tfrac{1}{2},$$

that is,

$$x = 2, \quad \text{and} \quad x = \tfrac{3}{2}.$$

These two values correspond to those, which were found above between $\frac{6}{3}$ and $\frac{5}{3}$, and between $\frac{5}{3}$ and $\frac{4}{3}$, and which differ from each other by a quantity less than unity.

In order to obtain the first, which answers to the supposition of $y = 1$, to a greater degree of exactness, we make

$$y = 1 + \frac{1}{y'},$$

we then have

$$y'^3 - 2y'^2 - y' + 1 = 0.$$

We find in this equation only one root greater than unity, and that is between 2 and 3, which gives

$$y = 1 + \tfrac{1}{2} = \tfrac{3}{2},$$

whence

$$x = 1 + \tfrac{2}{3} = \tfrac{5}{3}.$$

Again, if we suppose $y' = 2 + \dfrac{1}{y''}$, we shall be furnished with the equation

$$y''^3 - 3y''^2 - 4y'' - 1 = 0;$$

we find the value of y'' to be between 4 and 5; taking the smallest of these numbers, 4, we have

$$y' = 2 + \tfrac{1}{4}, \quad y = 1 + \tfrac{4}{9} = \tfrac{13}{9}, \quad x = 1 + \tfrac{9}{13} = \tfrac{22}{13}.$$

It would be easy to pursue this process, by making $y'' = 4 + \dfrac{1}{y'''}$, and so on.

I return now to the second value of x, which, by the first approximation, was found equal to $\tfrac{3}{2}$, and which answers to the supposition of $y = 2$. Making $y = 2 + \dfrac{1}{y'}$ and substituting this expression in the equation involving y, we have, after changing the signs in order to render the first term positive,

$$y'^3 + y'^2 - 2y' - 1 = 0.$$

This equation, like the corresponding one in the above operation, has only one root greater than unity, which is found between 1 and 2; taking $y' = 1$, we have

$$y = 3, \qquad x = \tfrac{4}{3}.$$

Again assuming

$$y' = 1 + \frac{1}{y''},$$

we are furnished with the equation

$$y''^3 - 3y''^2 - 4y'' - 1 = 0,$$

in which y'' is found to be between 4 and 5, whence

$$y' = \tfrac{5}{4}, \qquad y = \tfrac{14}{5}, \qquad x = \tfrac{19}{14}.$$

We may continue the process by making $y'' = 4 + \dfrac{1}{y'''}$ and so on.

The equation $x^3 - 7x + 7 = 0$ has also one negative root, between -3 and -4. In order to approach it more nearly, we make $x = -3 - \dfrac{1}{y}$; which gives

$$y^3 - 20y^2 - 9y - 1 = 0, \; y > 20 \text{ and } < 21,$$

whence

$$x = -3 - \tfrac{1}{20} = -\tfrac{61}{20}.$$

To proceed further, we may suppose $y = 20 + \frac{1}{y'}$ &c., we shall then obtain, successively, values more and more exact.

The several equations transformed into equations in y, y', y'', &c. will have only one root greater than unity, unless two or more roots of the proposed equation are comprehended within the same limits a and $a + 1$; when this is the case, as in the above example, we shall find in some one of the equations in y, y', &c. several values greater than unity. These values will introduce the different series of equations, which show the several roots of the proposed equation, that exist within the limits a and $a + 1$.

The learner may exercise himself upon the following equation;

$$x^3 - 2x - 5 = 0,$$

the real root of which is between 2 and 3; we find for the entire values of y, y', &c.

$$10,\ 1,\ 1,\ 2,\ 1,\ 3,\ 1,\ 1,\ 12,\ \&c.$$

and for the approximate values of x,

$$\tfrac{2}{1},\ \tfrac{21}{10},\ \tfrac{23}{11},\ \tfrac{44}{21},\ \tfrac{111}{53},\ \tfrac{155}{74},\ \tfrac{576}{275},\ \tfrac{731}{349},\ \tfrac{1307}{624},\ \tfrac{16415}{7837}.$$

Of Proportion and Progression.

223. Arithmetic introduces us to a knowledge of the definition and fundamental properties of *proportion* and *equidifference*, or of what is termed *geometrical* and *arithmetical proportion*. I now proceed to treat of the application of algebra to the principles there developed; this will lead to several results, of which frequent use is made in geometry.

I shall begin by observing, that equidifference and proportion may be expressed by equations. Let A, B, C, D, be the four terms of the former, and a, b, c, d, the four terms of the latter; we have then

$$B - A = D - C \text{ (Arith. 127)}, \quad \frac{b}{a} = \frac{d}{c} \text{ (Arith. 111)},$$

equations, which are to be regarded as equivalent to the expressions

$$A \,.\, B : C \,.\, D, \qquad a : b :: c : d,$$

and which give

$$A + D = B + C, \qquad a d = b c.$$

Hence it follows, that, *in equidifference, the sum of the extreme terms is equal to that of the means*, and, *in proportion, the product of the*

extremes is equal to the product of the means, as has been shown in Arithmetic (127, 113), by reasonings, of which the above equations are only a translation into algebraic expressions.

The reciprocal of each of the preceding propositions may be easily demonstrated; for from the equations

$$A + D = B + C, \qquad a\,d = b\,c,$$

we return at once to

$$D - C = B - A, \qquad \frac{b}{a} = \frac{d}{c},$$

and, consequently, *when four quantities are such, that two among them give the same sum, or the same product, as the other two, the first are the means and the second the extremes* (or the converse) *of an equidifference or proportion.*

When $B = C$, the equidifference is said to be *continued*; the same is said of proportion, when $b = c$. We have in this case

$$A + D = 2B, \qquad a\,d = b^2:$$

that is, *in continued equidifference, the sum of the extremes is equal to double the mean;* and *in proportion, the product of the extremes is equal to the square of the mean.* From this we deduce

$$B = \frac{A + D}{2}, \qquad b = \sqrt{a\,d};$$

the quantity B is the *middle* or mean arithmetical proportional between A and D, and the quantity b the *mean geometrical proportional* between a and d.

The fundamental equations,

$$B - A = D - C, \qquad \frac{b}{a} = \frac{d}{c},$$

lead also to the following;

$$C - A = D - B, \qquad \frac{c}{a} = \frac{d}{b};$$

from which it is evident, that we may change the relative places of the means in the expressions $A \,.\, B : C \,.\, D$, $a : b :: c : d$, and in this way obtain $A \,.\, C : B \,.\, D$, $a : c :: b : d$. In general, we may make any transposition of the terms, which is consistent with the equations

$$A + D = B + C \text{ and } a\,d = b\,c \quad (\textit{Arith. } 114.)$$

I have now done with equidifference, and shall proceed to consider proportion simply.

224. It is evident, that to the two members of the equation $\frac{b}{a} = \frac{d}{c}$ we may add the same quantity m, or subtract it from them; so that we have

$$\frac{b}{a} \pm m = \frac{d}{c} \pm m;$$

reducing the terms of each member to the same denominator, we obtain

$$\frac{b \pm ma}{a} = \frac{d \pm mc}{c},$$

an equation, which may assume the form

$$\frac{c}{a} = \frac{d \pm mc}{b \pm ma},$$

and may be reduced to the following proportion,

$$b \pm ma : d \pm mc :: a : c;$$

and as $\frac{c}{a} = \frac{d}{b}$, we have likewise

$$\frac{d \pm mc}{b \pm ma} = \frac{d}{b},$$

or $$b \pm ma : d \pm mc :: b : d.$$

These two proportions may be enunciated thus; *The first consequent plus or minus its antecedent taken a given number of times, is to the second consequent plus or minus its antecedent taken the same number of times, as the first term is to the third, or as the second is to the fourth.*

Taking the sums separately and comparing them together, and also the differences, we obtain

$$\frac{d + mc}{b + ma} = \frac{c}{a}, \quad \frac{d - mc}{b - ma} = \frac{c}{a},$$

whence we conclude

$$\frac{d + mc}{b + ma} = \frac{d - mc}{b - ma},$$

that is,

$$b + ma : d + mc :: b - ma : d - mc;$$

or rather, by changing the relative places of the means

$$b + ma : b - ma :: d + mc : d - mc;$$

and if we make $m = 1$, we have simply

$$b + a : b - a :: d + c : d - c,$$

which may be enunciated thus;

The sum of the first two terms is to their difference as the sum of the last two is to their difference.

225. The proportion $a : b :: c : d$ may be written thus;

$$a : c :: b : d;$$

we have then

$$\frac{c}{a} \pm m = \frac{d}{b} \pm m,$$

whence

$$\frac{c \pm m a}{a} = \frac{d \pm m b}{b},$$

and lastly,

$$c \pm m a : d \pm m b :: a : b \text{ or } :: c : d,$$

from which it follows, that *the second antecedent plus or minus the first taken a given number of times, is to the second consequent plus or minus the first taken the same number of times, as any one of the antecedents whatever is to its consequent.*

This proposition may also be deduced immediately from that given in the preceding article; for by changing the order of the means in the original proportion

$$a : b :: c : d,$$

and applying the proposition referred to, we obtain, successively,

$$a : c :: b : d,$$

$$c \pm m a : d \pm m b :: a : b \text{ or } :: c : d,$$

and denominating the letters, a, b, c, d, in this last proportion, according to the place they occupy in the original proportion, we may adopt the preceding enunciation.

Making $m = 1$, we obtain the proportions

$$c \pm a : d \pm b :: a : b$$
$$:: c : d,$$
$$c + a : c - a :: d + b : d - b;$$

whence it appears, that *the sum or difference of the antecedents is to the sum or difference of the consequents, as one antecedent is to its consequent,* and that *the sum of the antecedents is to their difference as that of the consequents is to their difference.*

In general, if we have

$$\frac{b}{a} = \frac{d}{c} = \frac{f}{e} = \frac{h}{g}, \text{ \&c.},$$

and make $\frac{b}{a} = q$, we shall have

$$\frac{d}{c} = q, \ \frac{f}{e} = q, \ \frac{h}{g} = q, \text{ \&c.},$$

which gives

$$b = a q, \ d = c q, \ f = e q, \ h = g q, \text{ \&c.}$$

then, by adding these equations member to member, we obtain

$$b + d + f + h = aq + cq + eq + gq,$$

or $$b + d + f + h = q(a + c + e + g).$$

whence it follows, that

$$\frac{b + d + f + h}{a + c + e + g} = q = \frac{b}{a}.$$

This result is enunciated thus; *in a series of equal ratios,*

$$a : b :: c : d :: e : f :: g : h, \&c.$$

the sum of any number whatever of antecedents is to the sum of a like number of consequents, as one antecedent is to its consequent.

226. If we have the two equations

$$\frac{b}{a} = \frac{d}{c}, \text{ and } \frac{f}{e} = \frac{h}{g},$$

and multiply the first members together and the second together, the result will be

$$\frac{bf}{ae} = \frac{dh}{cg};$$

an equation equivalent to the proportion

$$ae : bf :: cg : dh,$$

which may be obtained also by multiplying the several terms of the proportion

$$a : b :: c : d,$$

by the corresponding ones in the proportion

$$e : f :: g : h.$$

Two proportions multiplied thus term by term are said to be *multiplied in order*; and the products obtained in this way, are, as will be seen, proportional; the new ratios are the ratios *compounded* of the original ratios (*Arith.* 123).

It will be readily perceived also, that if we divide two proportions term by term, or in *order*, the result will be a proportion.

227. If we have $$\frac{b}{a} = \frac{d}{c},$$

we may deduce from it

$$\frac{b^m}{a^m} = \frac{d^m}{c^m},$$

which gives

$$a^m : b^m :: c^m : d^m;$$

whence it follows, that *the squares, the cubes, and, in general, the similar powers of four proportional quantities are also proportional.*

The same may be said of fractional powers, for, since

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$$\sqrt[m]{\frac{b}{a}} = \frac{\sqrt[m]{b}}{\sqrt[m]{a}},$$

and

$$\sqrt[m]{\frac{d}{c}} = \frac{\sqrt[m]{d}}{\sqrt[m]{c}};$$

therefore,

$$\frac{\sqrt[m]{b}}{\sqrt[m]{a}} = \frac{\sqrt[m]{d}}{\sqrt[m]{c}},$$

or

$$\sqrt[m]{a} : \sqrt[m]{b} :: \sqrt[m]{c} : \sqrt[m]{d},$$

if $a : b :: c : d$; that is, *the roots of the same degree of four proportional quantities, are also proportional.*

Such are the leading principles in the theory of proportion. This theory was invented for the purpose of discovering certain quantities by comparing them with others. Latin names were for a long time used to express the different changes or transformations, which a proportion admits of. We are beginning to relieve the memory of the mathematical student from so unnecessary a burden; and this parade of proportions might be entirely superseded by substituting the corresponding equations, which would give greater uniformity to our methods, and more precision to our ideas.

228. We pass from proportion to progression by an easy transition. After we have acquired the notion of three quantities in continued equidifference, the last of which exceeds the second, as much as this exceeds the first, we shall be able, without difficulty, to represent to ourselves an indefinite number of quantities, a, b, c, d, &c., such, that each shall exceed the preceding one, by the same quantity δ, so that

$$b = a + \delta,\ c = b + \delta,\ d = c + \delta,\ e = d + \delta, \&c.$$

A series of these quantities is written thus;

$$\div\, a \,.\, b \,.\, c \,.\, d \,.\, e \,.\, f, \&c.$$

and is termed an *arithmetical progression*; I have thought it proper, however, to change this denomination to that of *progression by differences*. (See *Arith.* art. 127, note.)

We may determine any term whatever of this progression, without employing the intermediate ones. In fact, if we substitute for b its value in the expression for c, we have

$$c = a + 2\delta;$$

by means of this last, we find

$$d = a + 3\delta, \text{ then } e = a + 4\delta,$$

and so on; whence it is evident, that representing by l the term, the place of which is denoted by n, we have

$$l = a + (n - 1)\delta.$$

Let there be, for example, the progression

$$\div\ 3 . 5 . 7 . 9 . 11 . 13 . 15 . 17, \&c.$$

here the first term $a = 3$, the difference or *ratio* $\delta = 2$; we find for the eighth term

$$3 + (8 - 1)\,2 = 17,$$

the same result, to which we arrive by calculating the several preceding terms.

The progression we have been considering is called *increasing*; by reversing the order, in which the terms are written, thus,

$$\div\ 17 . 15 . 13 . 11 . 9 . 7 . 5 . 3 . 1 . -1 . -3, \&c.$$

we form a *decreasing* progression. We may still find any term whatever by means of the formula $a + (n - 1)\delta$, observing only, that δ is to be considered as negative, since, in this case, we must subtract the difference from any particular term in order to obtain the following.

229. We may also, by a very simple process, determine the sum of any number whatever of terms in a progression by differences. This progression being represented by

$$\div\ a . b . c \ldots\ldots\ldots . i . k . l,$$

and S denoting the sum of all the terms, we have

$$S = a + b + c \ldots\ldots\ldots\ldots + i + k + l.$$

Reversing the order, in which the terms of the second member of this equation are written, we have still

$$S = l + k + i \ldots\ldots\ldots\ldots + c + b + a.$$

If we add together these equations, and unite the corresponding terms, we obtain

$$2S = (a+l) + (b+k) + (c+i) \ldots + (i+c) + (k+b) + (l+a);$$

but by the nature of the progression, we have, beginning with the first term,

$$a + \delta = b,\ b + \delta = c, \ldots\ldots i + \delta = k,\ k + \delta = l,$$

and, consequently, beginning with the last

$$l - \delta = k,\ k - \delta = i, \ldots\ldots c - \delta = b,\ b - \delta = a;$$

by adding the corresponding equations, we shall perceive at once, that

$$a + l = b + k = c + i, \text{ \&c.}$$

and, consequently, that

$$2S = n(a + l);$$

whence it follows, that

$$S = \frac{n(a+l)}{2}.$$

Applying this formula to the progression

$$\div 3 \,.\, 5 \,.\, 7 \,.\, 9 \text{ \&c.}$$

we find for the sum of the first eight terms

$$\frac{(3 + 17)\,8}{2} = 80.$$

230. The equation

$$l = a + (n - 1)\delta,$$

together with

$$S = \frac{(a + l)\,n}{2},$$

furnishes us with the means of finding any two of the five quantities, a, δ, n, l, and S, when the other three are known; I shall not stop to treat of the several cases, which may be presented.

231. From proportion is derived progression by *quotients* or *geometrical* progression, which consists of a series of terms, such, that the quotient arising from the division of one term by that which precedes it, is the same, from whatever part of the series the two terms are taken. The series

$$\div\!\!\div\; 2 : 6 : 18 : 54 : 162 : \text{\&c.}$$

$$\div\!\!\div\; 45 : 15 : 5 : \tfrac{5}{3} : \tfrac{5}{9} : \text{\&c.}$$

are progressions of this kind; the quotient or *ratio* is 3 in the first, and $\frac{1}{3}$ in the other; the first is increasing, and the second decreasing. Each of these progressions forms a series of equal ratios, and for this reason is written, as above.

Let

$$a, b, c, d, \ldots\ldots\ldots\ldots k, l,$$

be the terms of a progression by quotients; making $\frac{b}{a} = q$, we have by the nature of the progression,

$$q = \frac{b}{a} = \frac{c}{b} = \frac{d}{c} = \frac{e}{d} \cdots = \frac{l}{k},$$

or $\quad b = aq,\ c = bq,\ d = cq,\ e = dq, \ldots\ldots l = kq.$

Substituting, successively, the value of b in the expression for c, and the value of c in the expression for d, &c., we have

$$b = aq,\ c = aq^2,\ d = aq^3,\ e = aq^4, \ldots l = aq^{n-1},$$

taking n to represent the place of the term l, or the number of terms considered in the proposed progression.

By means of the formula $l = aq^{n-1}$ we may determine any term whatever, without making use of the several intermediate ones. The tenth term of the progression

$$\div\!\!\div\, 2 : 6 : 18 : \&c.,$$

for example, is equal to $2 \times 3^9 = 39366$,

232. We may also find the sum of any number of terms we please of the progression

$$\div\!\!\div\, a : b : c : d, \&c.$$

by adding together the equations

$$b = aq,\ c = bq,\ d = cq,\ e = dq, \ldots\ l = kq;$$

for the result will be

$$b + c + d + e \ldots + l = (a + b + c + d \ldots + k)\,q;$$

and representing by S the sum sought, we have

$$b + c + d + e \ldots + l = S - a,$$
$$a + b + c + d \ldots + k = S - l,$$

whence $$S - a = q\,(S - l),$$

and, consequently,

$$S = \frac{ql - a}{q - 1}\dagger.$$

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† The truth of this result may be rendered very evident, independently of analysis. If it were required, for example, to find the sum of the progression

$$\div\!\!\div\, 2 : 6 : 18 : 54 : 162,$$

multiplying by the ratio, we have

$$\div\!\!\div\, 6 : 18 : 54 : 162 : 486.$$

The first series being subtracted from this gives $486 - 2$, equal to so many times the first series, as is denoted by the ratio minus one, that is

$$2 + 6 + 18 + 54 + 162 = \frac{3 \times 162 - 2}{3 - 1}.$$

If we multiply by the ratio q the general series

$$\div\!\!\div\, a : b : c : d : e \ldots\ l,$$

we have $$\div\!\!\div\, aq : bq : cq : dq : eq \ldots\ lq.$$

Then, because $b = aq$, &c., the second series minus the first is $lq - a$, equal to so many times the first series, as is denoted by the ratio minus one.

Hence $$a + b + c + d + e \ldots + l = \frac{lq - a}{q - 1}.$$

In the above example, we find for the sum of the first ten terms of the progression

$$\div\; 2 : 6 : 18 : \&c.$$

$$\frac{2 \times 3^{10} - 2}{2} = 3^{10} - 1 = 59048.$$

233. The two equations,

$$l = a\,q^{n-1}, \quad S = \frac{q\,l - a}{q - 1},$$

comprehend the mutual relations, which exist among the five quantities, a, q, n, l, and S, in a progression by quotients, and enable us to find any two of these quantities, when the other three are given.

234. If we substitute $a\,q^{n-1}$ in the place of l, in the expression for S, we have

$$S = \frac{a\,(q^n - 1)}{q - 1}.$$

When q is a whole number, the quantity q^n will become greater and greater in proportion to the increased magnitude of the number n; and S may be made to exceed any quantity whatever, by assigning a proper value to n, that is, by taking a sufficient number of terms in the proposed progression. But if q is a fraction, represented by $\frac{1}{m}$, we have

$$S = \frac{a\left(\frac{1}{m^n} - 1\right)}{\frac{1}{m} - 1} \dagger = \frac{a\,m\left(1 - \frac{1}{m^n}\right)}{m - 1} = \frac{a\,m - \frac{a}{m^{n-1}}}{m - 1};$$

and it is evident, that as the number n becomes greater, the term $\frac{a}{m^{n-1}}$ will become smaller, and, consequently, the value of S will approach nearer and nearer to the quantity $\frac{a\,m}{m - 1}$, from which it will differ only by

$$\frac{a}{(m - 1)\,m^{n-1}};$$

therefore, the greater the number of terms we take in the proposed progression, the more nearly will their sum approach to

† Multiplying the numerator and denominator by $-\,m$.

$\frac{am}{m-1}$. It may even differ from $\frac{am}{m-1}$ by a quantity less than any assignable quantity, without ever becoming in a rigorous sense equal to it.

The quantity $\frac{am}{m-1}$, which I shall designate by L, forms, we perceive, a limit, to which the particular sums represented by S, approach nearer and nearer.

Applying what has been said to the progression

$$\div\!\div 1 : \tfrac{1}{2} : \tfrac{1}{4} : \tfrac{1}{8} : \tfrac{1}{16}, \text{ \&c.}$$

we have

$$a = 1,\; q = \frac{1}{m} = \tfrac{1}{2},$$

whence

$$m = 2,\quad L = \frac{am}{m-1} = 2;$$

and the greater the number of terms we take in the above progression, the nearer their sum will approach to an equality with 2.

We have, in fact

$$\begin{array}{lllll}
1 & = 1 & = 2 - 1, \\
1 + \frac{1}{2} & = \frac{3}{2} & = 2 - \frac{1}{2}, \\
1 + \frac{1}{2} + \frac{1}{4} & = \frac{7}{4} & = 2 - \frac{1}{4}, \\
1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} & = \frac{15}{8} & = 2 - \frac{1}{8}, \\
1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} & = \frac{31}{16} & = 2 - \frac{1}{16}.
\end{array}$$

&c.

The expression for L may be considered as the sum of the decreasing progression by quotients, continued to infinity, and it is thus, that it is usually presented; but in order to form a clear idea of it, we must represent it in a limited view.

235. We may obtain from the expression

$$S = \frac{a(q^n - 1)}{q - 1},$$

all the terms of the progression, of which it denotes the sum; for, if we divide $q^n - 1$ by $q - 1$ (158), we find

$$\frac{q^n - 1}{q - 1} = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + q^3 + q^4 \ldots\ldots + q^{n-1},$$

which gives

$$S = a + aq + aq^2 \ldots\ldots + aq^{n-1}.$$

We may employ the value of L for the same purpose; in this case, m is to be divided by $m - 1$, as follows;

$$\begin{array}{r|l} m & m - 1 \\ \cline{2-2} -m + 1 & 1 + \frac{1}{m} + \frac{1}{m^2} + \frac{1}{m^3} + \&c. \\ \cline{2-2} -1 + \frac{1}{m} & \\ \quad -\frac{1}{m} + \frac{1}{m^2} & \\ \qquad -\frac{1}{m^2} + \frac{1}{m^2} & \\ \&c. & \end{array}$$

We begin, by dividing, according to the usual method, by the first term, and find 1 for the quotient; we multiply this quotient by the divisor and subtract the product from the dividend; then, dividing the remainder by the first term of the divisor, we obtain $\frac{1}{m}$ for the quotient, and have $\frac{1}{m}$ for a remainder; we go through the same process with this remainder as with the preceding. Pursuing this method, we soon discover the law, to which the several particular quotients are subjected, and perceive that the expression $\frac{m}{m-1}$ is equivalent to the series

$$1 + \frac{1}{m} + \frac{1}{m^2} + \frac{1}{m^3} + \&c.$$

continued to infinity. Substituting for m its value $\frac{1}{q}$, and multiplying by a, we find as before

$$a + aq + aq^2 + aq^3 + \&c.$$

for the progression of which L represents the limit.

236. The series

$$1 + \frac{1}{m} + \frac{1}{m^2} + \frac{1}{m^3} + \&c.$$

is considered as the value of the fraction $\frac{m}{m-1}$, whenever it is *converging*, that is, when the terms, of which it is composed, become smaller and smaller the further they are removed from the first.

Indeed, if we make the division cease successively at the first, second, third . . . . . . remainder, we have

the quotients	and the remainders
1	1
$1+\frac{1}{m}$	$\frac{1}{m}$
$1+\frac{1}{m}+\frac{1}{m^2}$	$\frac{1}{m^2}$
&c.	&c.

the former of which approach the true value, exactly in proportion as the latter are diminished; and this takes place, only when m exceeds unity. In all other cases we must have regard to the remainders, which, increasing without limit, make it evident, that the quotients are departing further and further from the true value.

To render this clear, we have only to make, successively, $m = 2$, $m = 1$, $m = \frac{1}{2}$. Upon the first supposition, we have

$$\frac{m}{m-1} = 2 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \text{\&c.}$$

and it has been shown (234), that the series, which constitutes the second member, approaches, in fact, nearer and nearer to 2.

The second supposition leads us to

$$\frac{m}{m-1} = \tfrac{1}{0} = 1 + 1 + 1 + 1 + 1 + 1 + 1 \text{ \&c.}$$

This result, $1 + 1 + 1 + 1 + 1$ &c., continued to infinity, presents in reality an infinite quantity, as the nature of the expression $\frac{1}{0}$ implies; yet if we neglect the remainders in this example, we are led into an absurdity; for since the divisor, multiplied by the quotient, must produce the dividend, we have

$$1 = (1 + 1 + 1 + 1 + \ldots\ldots)\, 0;$$

but the second member is strictly reduced to nothing, we have therefore $1 = 0$.

The third supposition leads to consequences not less absurd, if we neglect the remainders, and consider the series, which is obtained, as expressing the value of the fraction, from which it is derived. Making $m = \frac{1}{2}$, we find

$$\frac{m}{m-1} = -1 = 1 + 2 + 4 + 8 + 16 + \text{\&c.},$$

which is evidently false.

There will be no contradiction of this kind, if we observe, that, in the second case, the remainders

$$1, \frac{1}{m}, \frac{1}{m^2}, \frac{1}{m^3}, \text{\&c.}$$

are each equal to 1, and that, since they do not diminish, they can never be neglected, to whatever extent the series is continued. If we add, therefore, one of these remainders to the second member of the equation

$$1 = (1 + 1 + 1 + 1 + 1 + \ldots\ldots)\,0,$$

the equation becomes true. In the third case, the remainders,

$$1, \frac{1}{m}, \frac{1}{m^2}, \frac{1}{m^3}, \&c.,$$

form the increasing progression, 1, 2, 4, 8, 16, &c. and, if we add to the several quotients the fractions, arising from the corresponding remainders, the exact expressions for $\frac{m}{m-1}$ will be

$$1 + \frac{1}{m-1}$$

$$1 + \frac{1}{m} + \frac{1}{m(m-1)}$$

$$1 + \frac{1}{m} + \frac{1}{m^2} + \frac{1}{m^2(m-1)}$$

&c.,

each of which gives -1, when $m = \frac{1}{2}$.

If we take $m = -n$, the fraction $\frac{m}{m-1}$ becomes $\frac{n}{n+1}$; the series, which is produced by developing this fraction, assumes the form

$$1 - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} + \&c.,$$

and making $n = 1$, we have

$$1 - 1 + 1 - 1 + 1 - 1 + \&c.,$$

a series, which becomes alternately 1 and 0, and which, consequently, as often exceeds, as it falls below, the true value of $\frac{n}{n+1}$, equal in this case to $\frac{1}{2}$; but as the above series is not converging, it cannot give this true value; and we must, therefore, take into consideration the remainder, at whatever term we stop.

If we suppose, in the preceding series, $n = 2$, we have

$$1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \tfrac{1}{16}, - \&c.,$$

a series, in which the particular sums, 1, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{8}$, &c. are, alternately, smaller and greater than the true value of $\frac{n}{n+1}$, which

is $\frac{1}{2}$, but to which they approach continually, because the proposed series is converging.

Although *diverging* series, that is, those, the terms of which go on increasing, continue to depart further and further from the true value of the expressions from which they are derived, yet considered as developments of these expressions, they may serve to show such of their properties, as do not depend on their summation.

237. If we continue any process of division in algebra, according to the method pursued above (235), with respect to the quantities m and $m - 1$, the quotient will always be expressed by an infinite series composed of *simple terms*. Infinite series are also formed by extracting the roots of imperfect powers, and continuing the operation upon the several successive remainders; but they are obtained more easily by means of the formula for binomial quantities, as will be shown in the *Supplement*, where I shall treat of the more common series.

Theory of Exponential Quantities and of Logarithms.

238. In the several questions, we have resolved thus far, the unknown quantities have not been made subjects of consideration as exponents; this will be requisite, however, if we would determine the number of terms in a progression by quotients, of which the first term, the last term, and the ratio are given. In fact, we are furnished by a question of this kind with the equation

$$l = a q^{n-1} \text{ (231)},$$

in which n will be the unknown quantity; abridging the expression, by making $n - 1 = x$, we have $l = a q^x$. This equation cannot be resolved by the direct methods hitherto explained; and quantities like x cannot be represented by any of the signs already employed. In order to present this subject in a more clear light, I shall go back to state, according to Euler, the connexion which exists between the several algebraic operations, and the manner, in which they give rise to new species of quantities.

239. Let a and b be two quantities, which it is required to add together; we have

$$a + b = c;$$

and in seeking a or b from this equation, we find

$$a = c - b, \quad b = c - a;$$

hence the origin of subtraction; but when this last operation cannot be performed in the order in which it is indicated, the result becomes negative.

The repeated addition of the same quantity gives rise to multiplication; a representing the multiplier, b the multiplicand, and c the product, we have

$$ab = c,$$

whence we obtain

$$a = \frac{c}{b}, \quad b = \frac{c}{a};$$

and hence arises division, and fractions, in which this division terminates, when it cannot be performed without a remainder.

The repeated multiplication of a quantity by itself produces the powers of this quantity; if b represent the number of times a is a factor in the power under consideration, we have

$$a^b = c.$$

This equation differs essentially from the preceding, as the quantities a and b do not both enter into it of the same form, and hence the equation cannot be resolved in the same way with respect to both. If it be required to find a, it may be obtained by simply extracting the root, and this operation gives rise to a new species of quantities, denominated irrational; but b must be determined by peculiar methods, which I shall proceed to illustrate, after having explained the leading properties of the equation $a^b = c$.

240. It is evident, that if we assign a constant value greater than unity to a, and suppose that of b to vary, as may be requisite, we may obtain successively for c all possible numbers. Making $b = 0$, we have $c = 1$; then since b increases, the corresponding values of c will exceed unity more and more, and may be rendered as great as we please. The contrary will be the case, if we suppose b negative; the equation $a^b = c$ being then changed into $a^{-b} = c$, or $\frac{1}{a^b} = c$, the values of c will go on decreasing, and may be rendered indefinitely small. We may, therefore, obtain from the same equation all possible positive numbers, whether entire or fractional, upon the supposition, that a exceeds unity. The same is true, if we have $a < 1$; only the order, in which the values stand, will be reversed; but if we suppose $a = 1$, we shall always find $c = 1$, whatever value be

assigned to b; we must, therefore, consider the observations which follow, as applying only to cases, in which a differs essentially from unity.

In order to express more clearly, that a has a constant value, and that the two other quantities b and c are indeterminate, I shall represent them by the letters x and y; we then have the equation $a^x = y$, in which each value of y answers to one value of x, so that either of these quantities may be determined by means of the other.

241. This fact, that all numbers may be produced by means of the powers of one, is very interesting, not only when considered in relation to algebra, but also on account of the facility with which it enables us to abridge numerical calculations. Indeed, if we take another number y', and designate by x' the corresponding value of x, we shall have $a^{x'} = y'$, and, consequently, if we multiply y by y', we have

$$y y' = a^x \times a^{x'} = a^{x+x'};$$

if we divide the same, the one by the other, we find

$$\frac{y'}{y} = \frac{a^{x'}}{a^x} = a^{x'-x};$$

lastly, if we take the m^{th} power of y, and the n^{th} root, we have

$$y^m = (a^x)^m = a^{mx}$$

for the one, and

$$y^{\frac{1}{n}} = (a^x)^{\frac{1}{n}} = a^{\frac{x}{n}}$$

for the other.

It follows from the first two results, that knowing the exponents x and x' belonging to the numbers y and y', we may, by taking their sum, find the exponent which answers to the product yy', and by taking their difference, that which answers to the quotient $\frac{y'}{y}$. From the last two equations it is evident, that the exponent belonging to the m^{th} power of y may be obtained by simple multiplication, and that which answers to the n^{th} root, by simple division.

Hence it is obvious, that by means of a table, in which against the several numbers y, are placed the corresponding values of x, y being given, we may find x, and the reverse; and the *multiplication of any two numbers is reduced to simple addition*, because, instead of employing these numbers in the operation, we may add the corresponding values of x, and then seeking in the table

the number, to which this sum answers, we obtain the product required. The quotient of the proposed numbers, may be found, in the same table, opposite the difference between the corresponding values of x, and, therefore, *division is performed by means of subtraction.*

These two examples will be sufficient to enable us to form an idea of the utility of tables of the kind here described, which have been applied to many other purposes since the time of Napier, by whom they were invented. The values of x are termed *logarithms*, and, consequently, *logarithms are the exponents of the powers, to which a constant number must be raised, in order that all possible numbers may be successively deduced from it.*

The constant number is called the base of the table or system of logarithms.

I shall, in future, represent the logarithm of y by $\mathrm{l}\,y$; we have then $x = \mathrm{l}\,y$, and since $y = a^x$, we are furnished with the equation $y = a^{\mathrm{l}y}$.

242. As the properties of logarithms are independent of any particular value of the number a, or of their base, we may form an infinite variety of different tables by giving to this number all possible values, except unity. Taking, for example, $a = 10$, we have $y = (10)^{\mathrm{l}y}$, and we discover at once, that the numbers

$$1,\quad 10,\quad 100,\quad 1000,\quad 10000,\quad 100000,\quad \&c.,$$

which are all powers of 10, have for logarithms, the numbers

$$0,\quad 1,\quad 2,\quad 3,\quad 4,\quad 5,\quad \&c.$$

The properties mentioned in the preceding article may be verified in this series; thus if we add together the logarithms of 10 and 1000, which are 1 and 3, we perceive, that their sum, 4, is found directly under 10000, which is the product of the proposed numbers.

243. The logarithms of the intermediate numbers, between 1 and 10, 10 and 100, 100 and 1000, &c. can be found only by approximation. To obtain, for example, the logarithm of 2, we must resolve the equation $(10)^x = 2$, by the method given in art. 221, finding first the entire number approaching nearest to the value of x. It is obvious at once, that x is between 0 and 1, since $(10)^0 = 1$, $(10)^1 = 10$; we make therefore $x = \frac{1}{z}$, the equation then becomes $(10)^{\frac{1}{z}} = 2$, or $10 = 2^z$; now z is found

between 3 and 4; we suppose, therefore, $z = 3 + \frac{1}{z'}$, and hence

$$10 = 2^{3+\frac{1}{z'}} = 2^3 \times 2^{\frac{1}{z'}} = 8 \times 2^{\frac{1}{z'}},$$

or $$2^{\frac{1}{z'}} = \tfrac{10}{8} = \tfrac{5}{4},$$

or, lastly, $$2 = (\tfrac{5}{4})^{z'}.$$

As the value of z' is between 3 and 4, we make

$$z' = 3 + \frac{1}{z''};$$

we have then

$$2 = (\tfrac{5}{4})^{3+\frac{1}{z''}} = (\tfrac{5}{4})^3 \cdot (\tfrac{5}{4})^{\frac{1}{z''}},$$

whence we obtain

$$(\tfrac{5}{4})^{\frac{1}{z''}} = 2\,(\tfrac{4}{5})^3 = \tfrac{128}{125}, \text{ or } (\tfrac{128}{125})^{z''} = \tfrac{5}{4};$$

and after a few trials we discover that z'' is between 9 and 10. The operation may be continued further; but as I have exhibited this process merely to show the possibility of finding the logarithms of all numbers, I shall confine myself to the supposition of $z'' = 9$; we have then, going back through the several steps,

$$z' = \tfrac{28}{9}, \quad z = \tfrac{93}{28}, \quad x = \tfrac{28}{93}.$$

This value of x, reduced to decimals, is exact to the fourth figure, as it gives

$$x = 0{,}30107.$$

By calculations carried to a greater degree of exactness, it is found, that

$$x = 0{,}3010300,$$

the decimal figures being extended to seven places.

Regarding this value of x as an exponent, we must conceive the number 10 to be raised to the power denoted by the number 3010300, and the root of the result to be taken for the degree denoted by 10000000; we thus arrive at a number approaching very nearly to 2; that is $(10)^{\frac{3010300}{10000000}} = 2$, very nearly; the first member is a little greater than 2; but $(10)^{\frac{3010299}{10000000}}$ is smaller.*

* The method explained in this article becomes impracticable, when the numbers, the logarithms of which are required, are large; another method however, which may be very useful, is given by Long, an English geometer, in the *Philosophical Transactions* for the year 1724, No. 339.

244. By multiplying the logarithm of 2, successively, by 2, 3, 4, &c., we obtain logarithms of the numbers, 4, 8, 16, &c., which are the 2^d^, 3^d^, 4^th^, &c. powers of 2.

By adding to the logarithm of 2 the logarithms of 10, 100, 1000, &c. we obtain those of 20, 200, 2000, &c., it is evident, therefore, that if we have the logarithms of the former numbers, we may find the logarithms of all numbers composed of them, which latter can be only powers or products of the former. The number 210, for example, being equal to

$$2 \times 3 \times 5 \times 7,$$

its logarithm is equal to

$$l\,2 + l\,3 + l\,5 + l\,7,$$

and since $5 = \frac{10}{2}$, we have

$$l\,5 = l\,10 - l\,2.$$

As the process for determining x in the equation $(10)^x = y$ is very laborous, we may, reversing the order, furnish ourselves with the several expressions for x, then forming a table of the values of y corresponding to those of x, we shall afterwards, as will be perceived, be able, in any particular case, to determine x by means of y.

We take first for x the values comprehended between 0,1 and 0,9; we have then only to determine the value of y, which answers to $x = 0{,}1$, or $(10)^{\frac{1}{10}}$, because the several other values of y, namely,

$$(10)^{\frac{2}{10}}, \quad (10)^{\frac{3}{10}}, \text{ \&c.}$$

are the 2^d^, 3^d^, &c. powers of the first.

By extracting the square root, we discover at once, that

$$(10)^{\frac{1}{2}} \quad \text{or} \quad (10)^{\frac{5}{10}} = 3{,}162277660;$$

then taking the fifth root of this result, we have

$$(10)^{\frac{1}{10}} = 1{,}258925412.$$

By a similar process, we deduce from

$$(10)^{\frac{1}{10}} = 1{,}258925412,$$

the value of

$$\sqrt{(10)^{\frac{1}{10}}} = (10)^{\frac{1}{20}} = (10)^{\frac{5}{100}} = 1{,}122018454;$$

then taking the fifth root, we have

$$(10)^{\frac{1}{100}} = 1{,}023292992;$$

and raising the result to the 2^d^, 3^d^, ... 9^th^ powers, we obtain the values of y, corresponding to those of x comprehended between 0,01 and 0,09.

It will be readily seen, that by this method, we may also find the

245. Logarithms, which are always expressed by decimals, are composed of two parts, namely, the units placed on the left of the comma, and the decimal figures found on the right. The

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values of y for those of x between 0,001 and 0,009, between 0,0001 and 0,0009; thus we shall be furnished with the following table.

Log.	Nat. Num.	Log.	Nat. Num.
0,9	7,943282347	0,00009	1,000207254
8	6,309573445	8	1,000184224
7	5,011872336	7	1,000161194
6	3,981071706	6	1,000138165
5	3,162277660	5	1,000115136
4	2,511886432	4	1,000092106
3	1,995262315	3	1,000069080
2	1,584893193	2	1,000046053
1	1,258925412	1	1,000023026
0,09	1,230268771	0,000009	1,000020724
8	1,202264435	8	1,000018421
7	1,174897555	7	1,000016118
6	1,148153621	6	1,000013816
5	1,122018454	5	1,000011513
4	1,096478196	4	1,000009210
3	1,071519305	3	1,000006908
2	1,047128548	2	1,000004605
1	1,023292992	1	1,000002302
0,009	1,020939484	0,0000009	1,000002072
8	1,018591388	8	1,000001842
7	1,016248694	7	1,000001611
6	1,013911386	6	1,000001381
5	1,011579454	5	1,000001151
4	1,009252886	4	1,000000921
3	1,006931669	3	1,000000690
2	1,004615794	2	1,000000460
1	1,002305238	1	1,000000230
0,0009	1,002074475	0,00000009	1,000000207
8	1,001843766	8	1,000000184
7	1,001613109	7	1,000000161
6	1,001382506	6	1,000000138
5	1,001151956	5	1,000000115
4	1,000921459	4	1,000000092
3	1,000691015	3	1,000000069
2	1,000460623	2	1,000000046
1	1,000230285	1	1,000000023

By means of this table, we may find the logarithm of any number whatever, by dividing it by 10 a sufficient number of times. To obtain, for example, the logarithm of 2549, we first divide this num-

first of these is called the *characteristic*, because in the logarithms under consideration, which are adapted to the supposition of $a = 10$, and which are called *common logarithms*, this part shows,

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ber by $(10)^3$ or 1000, which is the greatest power of 10 it contains; we have then

$$2549 = (10)^3 \times 2{,}549;$$

we then seek in the table the power of 10 immediately below 2,549, and find

$$(10)^{0,4} = 2{,}511886432;$$

dividing 2,549 by this last number, we have

$$2{,}549 = (10)^{0,4} \times 1{,}014775177,$$

Again seeking in the table the power of 10 immediately below 1,014775177 we find

$$(10)^{0006,} = 1{,}013911386;$$

then dividing the preceding quotient 1,014775177 by this number, we obtain a third quotient 1,000851742. This process is to be continued, until we arrive at a quotient, which differs from unity only in those decimal places we propose to neglect.

If we consider, in the present case, the third quotient as equal to unity, the proposed number will be resolved into factors, which will be powers of 10, for we shall have

$$2549 = (10)^3 \times (10)^{0,4} \times (10)^{0,006} = (10)^{8,406},$$

from which it is evident, that 3,406 is the logarithm of the number 2549. By extending the divisions to 7 in number, this logarithm will be found to be 3,406869.

The same table enables us with still more ease to find a number by means of its logarithm, as in the following example.

Let 2,547 be the given logarithm; the number sought will be

$$(10)^{2,547} = (10)^2 \times (10)^{0,5} \times (10\ )^{00,4} \times (10)^{0,007};$$

it will, therefore, be equal to the product of the numbers

$$\begin{aligned}(10)^2 &= 100,\\ (10)^{0,5} &= 3{,}162277660,\\ (10)^{0,04} &= 1{,}096478196,\\ (10)^{0,007} &= 1{,}016248694,\end{aligned}$$

taken from the table; and will, consequently, be

$$2{,}547 = 1\,.\,352{,}357.$$

A table of the same kind with the above, but much more extended, has been published in England, by Dodson, the object of which is to furnish the means of finding the number answering to a given logarithm.

to what order of units the number corresponding to the logarithm belongs. The several logarithms of the numbers between 1 and 10, as they are between 0 and 1, have, necessarily, 0 for their characteristic; those of the numbers between 10 and 100 have 1 for their characteristic; those of the numbers between 100 and 1000 have 2; in general, the characteristic of a logarithm contains as many units, as the proposed number has figures, minus one.

246. It is important also to remark, that the decimal part of the logarithms of numbers, which are decuple the one of the other, is the same; for example,

the logarithm of	54360	is	4,7352794,
	5436		3,7352794,
	543,6		2,7352794,
	54,36		1,7352794,
	5,436		0,7352794;

for, as each of these numbers is the quotient of that which precedes it, divided by 10, the logarithm of the one is found by taking an unit from the characteristic of that of the other (241,242).

247. According to what has been said in art. 240, the logarithms of fractional numbers are, upon our present hypothesis, negative; and we may easily deduce them from those of entire numbers, if we observe that a fraction represents the quotient arising from the division of the numerator by the denominator. When the numerator is less than the denominator, its logarithm is also less than that of the denominator, and, consequently, if we subtract the latter from the former, the result will be negative.

In order to obtain the logarithm of the fraction $\frac{1}{2}$, for example, we subtract from 0, which denotes the logarithms of 1, the fraction 0,3010300, which represents that of 2; the result is

$$-0,3010300.$$

If we subtract from 0 the number 1,3010300, which is the logarithm of 20, we have the logarithm of $\frac{1}{20}$, equal to

$$-1,3010300.$$

The logarithm of 3 being 0,4771213, that of $\frac{2}{3}$ will be

$$0,3010300 - 0,4771213 = -0,1760913.$$

248. It is evident from the manner in which the logarithms of fractions are obtained, that, considered independently of their signs, they belong (241) to the quotients, arising from the division of the denominator by the numerator, and, consequently, an-

swer to the number, by which it is necessary to divide unity in order to obtain the proposed fraction. Indeed, $\frac{2}{3}$, for example, may be exhibited under the form $\frac{1}{\frac{3}{2}}$, and $l\frac{3}{2} = l3 - l2 = 0,1760913$.

It would be inconvenient, in order to find the value of a fraction, to which a given negative logarithm belongs, to employ the number to which the same logarithm answers when positive, since it would be necessary to divide unity by this number; but if we subtract this logarithm from 1, 2, 3, &c. units, the remainder will be the logarithm of a number, which expresses the fraction sought, when reduced to decimals, since this subtraction answers to the division of the numbers, 10, 100, 1000, &c. by the number to which the proposed logarithm belongs.

Let there be, for example, — 0,3010300; if without regarding the sign, we take this logarithm from 1, or 1,0000000, the remainder 0, 6989700, being the logarithm of 5, shows, that the fraction sought is equal to 0,5, since we supposed unity to be composed of 10 parts.

If, in seeking the logarithm of a fraction, we conceive unity to be made up of 10, or 100, or 1000, &c. parts, or which amounts to the same thing, if we augment the characteristic of the logarithm of the numerator by a number of units sufficient to enable us to subtract that of the denominator from it, we obtain in this way a positive logarithm, which may be employed in the place of that indicated above.

In order to introduce uniformity into our calculations, we most frequently augment the characteristic of the logarithm of the numerator by 10 units. If we do this with respect to the fraction $\frac{2}{3}$, for example, we have

$$10,3010300 - 0,4771213 = 9,8239087.$$

It will be readily seen, that this logarithm exceeds the negative logarithm — 0,1760913 by 10 units, and that, consequently, whenever we add it to others, we introduce 10 units too much into the result; but the subtraction of these ten units is easily performed, and by performing it we effect at the same time the subtraction of 0,1760913. Let N be the number, to which we add the positive logarithm 9,8239087; the result of the operation will be represented by

$$N + 10 - 0,1760913;$$

and if we subtract 10, we have simply

$$N - 0{,}1760913.$$

According to the preceding observations, we cause addition to take the place of subtraction, by employing, instead of the number to be subtracted, its *arithmetical complement*, that is, what remains, when this number is subtracted from one of the numbers, 10, 100, 1000, &c., a result which is obtained by taking the units of the proposed number from 10 and the several other figures from 9. We add this complement to the number, from which the proposed logarithm is to be subtracted, and from the sum subtract an unit of the same order as the complement.

It is evident, that if the complement is repeated several times, we must subtract, after the addition, as many units of the same order with the complement, as there are in the number, by which it is multiplied; and, for the same reason, if several complements are employed, we must subtract for each an unit of the same order, or as many units as there are complements, if they are all of the same order.

Sometimes this subtraction cannot be effected; in this case, the result is the arithmetical complement of the logarithm of a fraction, and answers in the tables to the expression of this fraction reduced to decimals. If 10 units remain to be taken from the characteristic, as is most frequently the case, the result is the same as if we had multiplied by 10000000000, the numerator of the fraction sought, in order to render it divisible by the denominator; the characteristic of the logarithm of the quotient shows the highest order of the units contained in this quotient, considered with reference to those of the dividend. In 9,8239087, the characteristic 9 shows, that the quotient must have one figure less than the number, by which we have multiplied unity; and, consequently, if we separate 10 figures for decimals, the first significant figure on the left will be tenths; and we shall find only hundredths, thousandths, &c., for the numbers the arithmetical complements of which have 8, 7, &c. for their characteristics.

249. What has been said respecting the *system* of logarithms, in which $a = 10$, brings into view the general principles necessary for understanding the nature of the tables; for more particular information the learner is referred to the tables themselves, which usually contain the requisite instruction relating to their arrangement and the method of using them. I will merely

mention the tables of Callet, stereotype edition, and those of Borda, as very complete and very convenient.

250. If we have the logarithm of a number y for a particular value of a, or for a particular base, it is easy to obtain the logarithm of the same number in any other system. If we have $a^x = y$; for another base A, we have $A^X = y$, X being different from x; hence we deduce $A^X = a^x$. Taking the logarithms according to the system the base of which is a, we have

$$\mathrm{l}\,A^X = \mathrm{l}\,a^x;$$

now $\mathrm{l}\,a^x = x$ by hypothesis, and $\mathrm{l}\,A^X = X\,\mathrm{l}\,A$ (241); therefore, $X\,\mathrm{l}\,A = x$, or $X = \frac{x}{\mathrm{l}\,A}$; but if we consider A as a base, X will be the logarithm of y in the system founded on this base; if, therefore, we designate this last by $\mathrm{L}\,y$, in order to distinguish it from the other, we have

$$\mathrm{L}\,y = \frac{\mathrm{l}\,y}{\mathrm{l}\,A},$$

and *we find the logarithm of* y *in the second system, by dividing its logarithm taken in the first by the logarithm of the base of the second system.*

The preceding equation gives also $\frac{\mathrm{l}\,y}{\mathrm{L}\,y} = \mathrm{l}\,A$; from which it is evident, that whatever be the number y, there is between the logarithms $\mathrm{l}\,y$ and $\mathrm{L}\,y$, a ratio invariably represented by $\mathrm{l}\,A$.

251. In every system the logarithm of 1 is always 0, since whatever be the value of a we have always $a^0 = 1$. As logarithms may go on increasing indefinitely, they are said to become infinite at the same time with the corresponding numbers; and as, when y is a fractional number, we have $y = \frac{1}{a^x} = a^{-x}$, it is evident, that in proportion as y becomes smaller, x in its negative state becomes greater, but we can never assign for x a number, which shall render y strictly nothing. In this sense it is said, that *the logarithm of zero is equal to an infinite negative quantity*, as we find in many tables.

252. I now proceed to give some examples of the use, which may be made of logarithms in finding the numerical value of formulas. It follows from what is said in art. 241, and from the definition of logarithms, by which we are furnished with the equation $a^{\mathrm{l}y} = y$, that

$$1(AB) = 1A + 1B, \qquad 1\left(\frac{A}{B}\right) = 1A - 1B,$$

$$1A^m = m\,1A, \qquad 1A^{\frac{1}{n}} = \frac{1}{n}\,1A.$$

Applying these principles to the formula

$$\frac{A^2\sqrt{B^2 - C^2}}{C\sqrt[5]{D^3\,EF}},$$

which is very complicated, we find

$$1(A^2\sqrt{B^2 - C^2}) = 1[A^2\sqrt{(B + C)(B - C)}] =$$
$$2\,1A + \tfrac{1}{2}\,1(B + C) + \tfrac{1}{2}\,1(B - C),$$

$$1(C\sqrt[5]{D^3\,EF}) = 1C + \tfrac{3}{5}\,1D + \tfrac{1}{5}\,1E + \tfrac{1}{5}\,1F,$$

and, consequently,

$$1\left(\frac{A^2\sqrt{B^2 - C^2}}{C\sqrt[5]{D^3\,EF}}\right) =$$

$$2\,1A + \tfrac{1}{2}\,1(B + C) + \tfrac{1}{2}\,1(B - C) - l\,C - \tfrac{3}{5}\,1D - \tfrac{1}{5}\,1E - \tfrac{1}{5}\,1F.$$

If we take the arithmetical complements of $1\,C$, $\frac{3}{5}\,1D$, $\frac{1}{5}\,1E$, $\frac{1}{5}\,1F$, designating them by C', D', E', F', instead of the preceding result, we have

$$2\,1A + \tfrac{1}{2}\,1(B + C) + \tfrac{1}{2}\,1(B - C) + C' + D' + E' + F',$$

only we must observe to subtract from the sum as many units of the same order with the complements, as there are complements taken, that is 4. When we have found the logarithm of the proposed formula, the tables will show the number, to which this logarithm belongs, which will be the value sought.

253. Logarithms are of most frequent use in finding the fourth term of a proportion. It is evident, that if $a : b :: c : d$ we have

$$d = \frac{bc}{a}, \quad \text{whence} \quad 1d = 1b + 1c - 1a;$$

that is, *the logarithm of the fourth term sought is equal to the sum of the logarithms of the two means, diminished by the logarithm of the known extreme*, or rather, *to the sum of the logarithms of the means, plus the arithmetical complement of the logarithm of the known extreme.*

254. If we take the logarithms of each member of the equation $\frac{b}{a} = \frac{d}{c}$, which presents the character of a proportion, we have

$$1b - 1a = 1d - 1c \ (252);$$

whence it follows, that the four logarithms

$$\mathrm{l}\,a \,.\, \mathrm{l}\,b : \mathrm{l}\,c \,.\, \mathrm{l}\,d$$

form an equidifference (223.

The series of equations,

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c} = \frac{e}{d}, \text{ \&c. } (231)$$

leads also to

$$\mathrm{l}\,b - \mathrm{l}\,a = \mathrm{l}\,c - \mathrm{l}\,b = \mathrm{l}\,d - \mathrm{l}\,c = \mathrm{l}\,e - \mathrm{l}\,d, \text{ \&c.},$$

and hence we infer, that the progression by quotients,

$$\div\!\!\div\, a : b : c : d : e, \text{ \&c.}$$

corresponds to the progression by differences,

$$\div\, \mathrm{l}\,a \,.\, \mathrm{l}\,b \,.\, \mathrm{l}\,c \,.\, \mathrm{l}\,d \,.\, \mathrm{l}\,e, \text{ \&c.},$$

and, consequently, *the logarithms of numbers in progression by quotients, form a progression by differences.*

255. If we have the equation $b^x = c$, we may easily resolve it by means of logarithms; for as $\mathrm{l}\,b^x$ is equal to $z\,\mathrm{l}\,b$, we have $z\,\mathrm{l}\,b = \mathrm{l}\,c$, and, consequently, $z = \frac{\mathrm{l}\,c}{\mathrm{l}\,b}$. The equation $b^{c^z} = d$ may be resolved in the same manner; making $c^x = u$, we have

$$b^u = d, \quad u\,\mathrm{l}\,b = \mathrm{l}\,d, \quad u = \frac{\mathrm{l}\,d}{\mathrm{l}\,b}, \quad \text{or} \quad c^x = \frac{\mathrm{l}\,d}{\mathrm{l}\,b};$$

again taking the logarithms, we find

$$z\,\mathrm{l}\,c = \mathrm{l}\left(\frac{\mathrm{l}\,d}{\mathrm{l}\,b}\right) = \mathrm{l}\,\mathrm{l}\,d - \mathrm{l}\,\mathrm{l}\,b \quad \text{and} \quad z = \frac{\mathrm{l}\,\mathrm{l}\,d - \mathrm{l}\,\mathrm{l}\,b}{\mathrm{l}\,c}.$$

In this last expression, $\mathrm{l}\,\mathrm{l}\,b$ represents the logarithm of the logarithm of b, and is found by considering this logarithm as a number. The quantities, b^x, b^{c^z}, and all which are derived from them, are called *exponential quantities.*

Questions relating to the Interest of Money.

256. The principles of progression by quotients and of logarithms will be found to occur in the calculations relating to interest. To understand what I have to offer on this subject it must be recollected, that the income derived from a sum of money employed in trade, or in executing some productive work will be in proportion to the frequency with which it is exchanged in either case. Hence it follows, that he, who borrows a sum of money for any purpose, ought, upon returning this money at the

expiration of a given time, to allow the lender a premium equivalent to the profits, which he might have received, if he had employed it himself. Such is the view in which the subject of interest presents itself. In order to determine the interest of any sum, we compare this sum with 100 dollars taken as unity, having fixed the premium, which ought to be allowed for this last at the end of a particular term, one year for example. I shall not here consider those things, which in the different kinds of speculation, occasion the rise and fall of interest; this belongs to the elements of political and commercial arithmetic, which should be preceded by some account of the doctrine of chances. My object in what follows is simply to resolve certain questions, which refer themselves to progression by quotients.

To present the subject in a general point of view, I shall suppose the annual premium, allowed for a sum 1, to be represented by r, r being a fraction; it is evident, that the interest of a sum 100, for the same time, will be $100\,r$, that of any sum whatever a will be denoted by $a\,r$; if we designate this last by α, we have

$$\alpha = a\,r.$$

By means of this formula, it is easy to find the interest of any sum whatever, when that of 100 or of any other sum, for a known time, is given; questions of this kind belong to what is called *simple interest.*

257. But if the lender, instead of receiving annually the interest of his money, leaves it in the hands of the borrower to accumulate, together with the original sum, during the following year, the value of the whole at the end of this year may be found in the following manner. The original sum being a, if we add to it the interest $a\,r$, it becomes at the end of the first year

$$a + a\,r = a\,(1 + r),$$

Now if we make

$$a\,(1 + r) = a',$$

the interest of the sum a' for one year being $a'\,r$, that of the sum $a(1 + r)$ will be, for a second year, $a\,r\,(1 + r)$; and as, at the end of the first year, the principal a, augmented by the interest, becomes $a\,(1 + r)$, the principal a' amounts, at the end of the second year, to

$$a'\,(1 + r) = a\,(1 + r)^2 = a''.$$

If the lender does not now withdraw the sum a'', but leaves it to accumulate during a third year, at the end of this, it will become, according to what precedes,

$$a''(1 + r) = a(1 + r)^3 = a'''.$$

It will be readily perceived, that a''' will become at the end of the fourth year

$$a'''(1 + r) = a(1 + r)^4,$$

and so on; and that, consequently, the sum first lent, and the several sums due at the end of the first, second, third, fourth, &c. years, form the following progression by quotients;

$$\div\!\!\div a : a(1 + r) : a(1 + r)^2 : a(1 + r)^3 : a(1 + r)^4 : \&c.$$

of which the quotient is $1 + r$, and the general term

$$a(1 + r)^n = A,$$

the number n representing the number of years, during which the interest is suffered to accumulate.

If the rate of interest be 5 per cent., for example, that is, if for 100 dollars during one year 105 dollars are paid back; we have

$$100\,r = 5, \quad \text{or} \quad r = \tfrac{5}{100} = \tfrac{1}{20}, \quad \text{and} \quad 1 + r = \tfrac{21}{20}.$$

If we would know to what the sum a amounts, when left to accumulate during 25 years, we have

$$n = 25, \quad \text{and} \quad a\left(\frac{21}{20}\right)^{25}$$

instead of the original sum. The 25th power of $\frac{21}{20}$ may be easily found by means of logarithms, since we have (252)

$$l\left(\frac{21}{20}\right)^{25} = 25\, l\,\tfrac{21}{20} = 25\,(l\,21 - l\,20) = 0{,}5297322,$$

which gives

$$\left(\frac{21}{20}\right)^{25} = 3{,}386 \quad \text{nearly}, \quad A = 3{,}386\,a;$$

and hence it may be readily seen, that 1000 dollars will in this way amount at compound interest to 3386 dollars, at the end of 25 years.

If the sum lent were for 100 years, we should have

$$A = a\left(\frac{21}{20}\right)^{100} = 131\,a$$

nearly; thus 1000 dollars would produce, at the end of this period, a sum of 131000 dollars nearly. These examples will be

sufficient to show with what rapidity sums accumulate by means of compound interest.

258. The equation

$$A = a(1 + r)^n,$$

gives rise to four questions; the first, which is to find A, when a, r, and n, are known, presents itself, whenever we seek the amount of the principal at the end of a number n of years. I have already given an example of this.

The second, which is to find r, when a, A, and n, are known, occurs whenever it is required to determine the rate of interest by means of the original sum, the whole amount that has become due, and the time during which it has been accumulating; we have in this case

$$1 + r = \sqrt[n]{\frac{A}{a}}.$$

The third, which is to find a, when A, r, and n are known, the formula for which is

$$a = \frac{A}{(1 + r)^n},$$

has for its object to determine the principal, which it is necessary to employ in order to be entitled, after a number n of years, to a sum A.

The fourth, which is to find n, when A, a, and r are known, can be resolved only by means of logarithms (238, 252). Taking the logarithm of each member of the proposed equation, we have

$$lA = \mathrm{l}a + n\,\mathrm{l}(1 + r),$$

whence

$$n = \frac{\mathrm{l}A - \mathrm{l}a}{\mathrm{l}(1 + r)}.$$

By means of this last equation we determine how many years the principal a must remain at interest in order to amount to a sum A.

To illustrate this by an example, I shall suppose that it is required to find the time in which the original sum will be doubled, the rate of interest being 5 per cent.; we have

$$A = 2a, \quad \mathrm{l}A = \mathrm{l}a + \mathrm{l}\,2,$$

and, consequently,

$$n = \frac{\mathrm{l}\,2}{\mathrm{l}\frac{21}{20}} = \frac{\mathrm{l}\,2}{\mathrm{l}\,21 - \mathrm{l}\,20} = \frac{0{,}3010300}{0{,}0211893} = 14{,}21,$$

nearly.

259. The following question is one of the most complicated,

that we meet with relating to this subject. We suppose, that the lender during a number n of years, adds each year a new sum, to the amount of this year; it is required to find what will be the value of these several sums, together with the compound interest that may thence arise at the expiration of the term proposed. Let a, b, c, d, ... k, be the sums added the first, second, third, fourth, &c. years; the sum a remaining in the hands of the borrower during a number n of years, amounts to

$$a(1+r)^n;$$

the sum b, which remains $n-1$ years only, becomes

$$b(1+r)^{n-1},$$

the sum c, which remains $n-2$ years only, becomes

$$c(1+r)^{n-2},$$

and so on; the last sum, k, which is employed only one year, becomes simply

$$k(1+r);$$

we have, therefore,

$$A = a(1+r)^n + b(1+r)^{n-1} + c(1+r)^{n-2} \ldots\ldots + k(1+r).$$

By calculating the several terms of the second member separately, we obtain the value of A.

The operation is very much simplified when

$$a = b = c = d \ldots\ldots = k,$$

for in this case we have

$$A = a(1+r)^n + a(1+r)^{n-1} + a(1+r)^{n-2} \ldots\ldots + a(1+r);$$

the second member of this equation forms a progression by quotients, of which the first term is $a(1+r)$, the last term $a(1+r)^n$, the quotient $1+r$, and the sum, consequently,

$$\frac{a(1+r)^{n+1} - a(1+r)}{r} \quad (232);$$

we have, therefore, in this case,

$$A = \frac{a(1+r)[(1+r)^n - 1]}{r}.$$

This equation gives rise also to four questions corresponding to those mentioned in connexion with the equation

$$A = a(1+r)^n.$$

260. By reversing the case we have been considering, we may represent those annual sums, or sums due at stated intervals, called *annuities;* here the borrower discharges a debt with the interest due upon it, by different payments made at regular periods. These payments, made by the borrower before the debt

in question is discharged, may be considered, as sums advanced to the lender toward the discharge of the debt, the value of which sums will depend upon the interval of time between the payment and the expiration of the annuity. Thus, if we represent each sum by a, the first payment, which will take place $n - 1$ years before the expiration of the term of the annuity, referred to this time, is worth $a(1 + r)^{n-1}$; the second, referred to the same epoch, is worth only $a(1 + r)^{n-2}$; the third, $a(1 + r)^{n-3}$, and so on to the last, which amounts only to the value of a. But on the other hand, the sum lent being represented by A, will be worth in the hands of the borrower, after n years, $A(1 + r)^n$, which must be equal to the amount of the several payments advanced by him to the lender; we have, therefore,
$A(1 + r)^n = a(1 + r)^{n-1} + a(1 + r)^{n-2} + a(1 + r)^{n-3} \ldots + a,$
or taking the sum of the progression, which constitutes the second member

$$A(1 + r)^n = \frac{a[(1 + r)^n - 1]}{r},$$

an equation, in which we may take for the unknown quantity, successively, the quantity A, which I shall call the *value* of the annuity, because it is the sum, which it represents, the quantity a, which is the *quota* of the annuity, the quantity r, which is the rate of interest, and lastly, the quantity n, which denotes the term of the annuity. In order to find this last we must have recourse to logarithms. We first disengage $(1 + r)^n$, which gives

$$(1 + r)^n = \frac{a}{a - Ar},$$

then taking the logarithms, we have

$$n\,\mathrm{l}(1 + r) = \mathrm{l}\,a - \mathrm{l}\,(a - A\,r),$$

whence

$$n = \frac{\mathrm{l}\,a - \mathrm{l}\,(a - A\,r)}{\mathrm{l}\,(1 + r)}.$$

261. To give an instance of the application of the above formulas, I shall take the following question;

To find what sum must be paid annually to cancel in 12 *years a debt of* 100 *dolls. with the interest during that time, the rate of interest being* 5 *per cent.*

In this example the quantities given are

$$A = 100, \quad n = 12, \quad r = \frac{1}{20},$$

and the annuity a is required to be found; resolving the equation

$$A(1+r)^n = \frac{a[(1+r)^n - 1]}{r}$$

with reference to the letter a, we have

$$a = \frac{Ar(1+r)^n}{(1+r)^n - 1}.$$

The values of the letters, A, r, and n, are to be substituted in this expression; and it will be found most convenient in the first place to calculate, by the help of logarithms, the quantity $(1+r)^n$, which becomes $(\frac{21}{20})^{12}$; and

$$(\tfrac{21}{20})^{12} = 1{,}79586.$$

By means of this value we obtain

$$a = \frac{100 \,.\, \frac{1}{20} \,.\, 1{,}79586}{1{,}79586 - 1} = \frac{5 \,.\, 1{,}79586}{0{,}79586};$$

and, determining the values of this last expression either directly or by means of logarithms, we find

$$a = 11{,}2826;$$

an annuity of 11,28 dolls., therefore, is necessary to cancel in 12 years a debt of 100 dolls., the rate of interest being 5 per cent.

262. I am prevented from entering into further details on this subject by the limits I have prescribed myself in this treatise; I will merely add, therefore, that in order to compare the values of different sums, as they concern the person, who pays or receives them, they must be reduced to the same epoch, that is, we must find what they would amount to when referred to the same date. A banker, for instance, owes a sum a payable in n years; as an equivalent he gives a note, the nominal value of which is represented by b, and which is payable in p years, the first sum at the time the note is given, is worth only $\frac{a}{(1+r)^n}$, because it must be be considered as the original value of a principal, which amounts to a at the expiration of n years; the sum b, for the same reason is worth at the time the note is given $\frac{b}{(1+r)^p}$; the difference

$$\frac{a}{(1+r)^n} - \frac{b}{(1+r)^p}$$

represents, therefore, according as it is positive or negative, what the banker ought to give or receive by way of balance; if this

balance is not to be paid until after a number of years denoted by q, c representing its value at the time the exchange is made, it will amount at the expiration of this term, to

$$c(1+r)^q;$$

so that it will be equivalent to

$$\left(\frac{a}{(1+r)^n} - \frac{b}{(1+r)^p}\right)(1+r)^q = a(1+r)^{q-n} - b(1+r)^{q-p}.$$

The several sums, a, b, ... k, in art. 259, were reduced to the time of the payment of the sum A, and in art. 260, each of the payments, as well as the sum A, was referred to the time, when the annuity was to cease.

[illegible] of the

[illegible] of the quantity [illegible]

[illegible] of which

[illegible]

[illegible]

[illegible] divisor [illegible] to the remainder

[illegible] greatest [illegible]

[illegible] smaller [illegible]

[illegible] $\times Q +$ [illegible]

[illegible] are divisible by a

[illegible] Q [illegible] D [illegible]

[illegible] $+$ [illegible] Q we get

[illegible] which is impossible

[illegible] number

[illegible] be infinity

[illegible] it is evident that [illegible]

[illegible] the divisor [illegible]

[illegible] indefinitely great

[illegible] can be taken from 0

[illegible] equal to infin-

NOTES.

(*Referred to Page* 81.)

In articles 66 and 75 I have interpreted the negative solutions by the examination of the equation, which they immediately verify, as I had done before, and this method appeared to me always exact, as the object is merely to show, that these solutions have a rational sense, since they resolve questions analagous to the one proposed; but there are often several ways of forming these questions, and the following, which was communicated to me by M. Français, a distinguished geometer, Professor at the School of Artillery of Mayence, seemed to me more simple, than that given in these Elements.

"He thinks, that we ought to leave out of the enunciation of the question of art. 65 the idea of the departure of the couriers, and to suppose them to have been travelling from an indefinite time; the question then would be stated thus. *Two couriers travel the same route in the same direction* C′ A B C (page 72); *after they have proceeded, each a certain time, one finds himself in* A *at the instant that the other is in* B; *their distance and rate of going are known; it is asked, at what point of the route they will encounter each other?*

This enunciation leads to the same equation, as that of art. 65; but "the continuity of the motion being once established, the negative solution admits of an explanation without the necessity of changing the direction of one of the couriers. Indeed, since their motion does not commence at the points A and B, but both, before arriving at these points, are supposed to have been going in the same manner for an indefinite time from C' toward B, it is easy to conceive, that the courier, who at this point is in advance of the one at A, who travels slower, must at a certain time have been behind him and overtaken him before his arrival at the point A. The sign — then indicates (as in the application of Algebra to Geometry) that the distance AR' is to be taken in a direction opposite to AR, which is regarded as positive. The change to be made in the enunciation, to render the negative solution positive, is reduced to supposing, that the two

couriers must have come together before their arrival at the point A, instead of its taking place afterward."

Indeed, when we place the point R' between A and C, instead of putting it between A and B, we find $AB = BR' - AR'$, whence results the equation $y - x = a$, instead of $x - y = a$, which we first obtained; and there is no need of changing the sign of c, the second equation remaining $\frac{x}{b} = \frac{y}{c}$.

M. Français applies not less happily these considerations to the case of art. 75, by substituting, for the couriers, moveable bodies, subjected to a continued motion commencing from an indefinite time. He enunciates the problem thus; "*Two moveable bodies are carried uniformly in a straight line* CB (page 80) *one in the direction* BC, *and the other in the direction* CB *with given velocities; that, which is carried in the first direction, is found in* B, *a known number of hours before the other has arrived at* A; *it is asked, at what point of the indefinite straight line* BC *their meeting takes place?*

The solution $x = + 48^{\text{mls.}}$ implies, that the two moveable bodies met at the point R, before that, which is carried from C towards B, had reached the point A, and that the second, which moves from B toward C, was at the point C, where he is found when the other is at the point A."

The position assigned to the point R, verifies itself by observing, that there results from it $AC = BC - AB = cd - a$, instead of $a + cd$, as first obtained (page 80,) and, conseqently,

$$\frac{x}{b} = \frac{cd - a - x}{c},$$

an equation which gives $x - 48$.

In this manner there is no change to be made in the direction of the motion; indeed there is a difference in the circumstances of the problem, and as I said before, this proves, that there are several physical questions corresponding to the same mathematical relations. But the enunciations, here given, have the advantage of not breaking the law of continuity, and this is derived from the consideration of lines, which represent in a manner the most simple and general, the circumstances of a change of sign in magnitudes. (See the *Elementary Treatise of Trigonometry and Application of Algebra to Geometry.*)

(Note referred to Page 185.)

It may be thought, that, in order to discover the roots of any equation of the fourth degree

$$x^4 + p x^3 + q x^2 + r x + s = 0,$$

it would be sufficient to compare it with the product of article 183, observing to put equal to each other the quantities by which the same power of x is multiplied; and it is in this manner that most elementary writers think to demonstrate, that *an equation of any degree whatever is the product of as many simple factors, as there are units in the exponent of its degree.* It will be seen by what follows, that the reasoning by which this is attempted to be proved, is defective. We stated the proposition with qualification in article 182, because it is necessary, in order to establish it unconditionally, to show that an equation of whatever degree has a root, real or imaginary, which is not easily done in an elementary work, and which happily is not necessary. Some remarks relating to this subject may be found in the *Supplement.*

By forming the equations,

$$\begin{aligned} -a-b-c-d &= p, \\ ab+ac+ad+bc+bd+cd &= q, \\ -abc-abd-acd-bcd &= r, \\ abcd &= s, \end{aligned}$$

in order to deduce from them the value of the letters, a, b, c, d, the roots of the proposed equation, the calculation would be very complicated, if, in the determination of the unknown quantities, a, b, c, d, we adopt the method of article 78; but if we multiply the first of the above equations by a^3, the second by a^2, the third by a, and add these three products to the fourth, member to member, we shall have

$$-a^4 = p a^3 + q a^2 + r a + s,$$

from which we derive, by simple transposition,

$$a^4 + p a^3 + q a^2 + r a + s = 0.$$

This equation contains only a, but it is entirely similar to the one proposed. The difficulty of obtaining a, therefore, is the same as that of obtaining x.

"Thus," says Castillon (Mém. de Berlin, année 1789) "it is shown in every work on algebra, that an equation of any degree we please, is formed of several simple binomials, but it is not so evident that an equation, formed by the multiplication of several simple binomials, can have such coefficients as we please."

If, instead of multiplying the first three equations in a, b, c, d, by a^3, a^2, and a, respectively, we multiply them by b^3, b^2, and b, or by c^3, c^2, c, or d^3, d^2, d, and add the products to the fourth equation, we shall have in the first

$$-b^4 = pb^3 + qb^2 + rb + s,$$

in the second

$$-c^4 = pc^3 + qc^2 + rc + s,$$

in the third

$$-d^4 = pd^3 + qd^2 + rd + s;$$

from which it follows, that we are conducted to the same equation in the case of a, in that of b, &c. Indeed the quantities, a, b, c, d, being all disposed in the same manner in each equation, it is not to be supposed that one should be determined by a different operation from that of the others; and, in general, if in the investigation of several unknown quantities, we are obliged to employ for each the same reasonings, the same operations, and the same known quantities, all these quantities will necessarily be roots of the same equation.

x + y – 9

QUESTIONS FOR PRACTICE

IN

LACROIX'S ALGEBRA.

I. *Addition, Art.* 18.

1. Add the quantities $x + yz + 42 - 29x - yz - 9$.
Ans. $33 - 28x$.

2. Add $147a + 23b - a - b + 2a$.
Ans. $148a + 22b$.

3. Add $11a + 11ab + 11abc - 11a + ab - 11abc$.
Ans. $12ab$.

4. Add $43a - 27c - 20a + 7c - 61b - 21a + 57b + 20c$.
Ans. $2a - 4b$.

5. Add $a + 9d + a - 7c + 8x - a + 6d + 6c - 7x - 14d$.
Ans. $a - c + d + x$.

6. Add $7abc + 6ab + 5c - abc + 21x + 9c - 27xy + 8abc + 10x - 93z + 10b + 31x - 2ab - c + 5xy - abc + 33z$.
Ans. $13abc + 4ab + 13c + 10b + 62x - 22xy - 60z$.

II. *Subtraction, Art.* 20.

7. From $6x - 8y + 3$
subtract $2x + 9y - 2$. *Ans.* $4x - 17y + 5$.

8. From $5xy - 8$
subtract $-3xy + 1$. *Ans.* $8xy - 9$.

9. From $4xy - x + xy$
subtract $2xy + 2 + xy$. *Ans.* $2xy - x - 2$.

10. From $5x + x - 8 - 4b$
subtract $6x - 10 + 4b - x$. *Ans.* $2 + x - 8b$.

11. From $148a + 47ab - 23abc + 1 - x$
subtract $99a - 47ab - 8abc - 2 + 4x$.
Ans. $49a + 94ab - 15abc + 3 - 5x$.

12. From $7b - 8c + 326xy - 43b + 111c + a$
subtract $-500b - 22a - 87xy - 7c$.
Ans. $464b + 23a + 110c + 413xy$.

III. *Multiplication, Art.* 32.

13. Multiply $12ax$ by $3a$. *Ans.* $36a^2x$.

14. Multiply $3xy - 8 + 2xyz$ by xy.
Ans. $3x^2y^2 - 8xy + 2x^2y^2z$.

15. Multiply $12x^2 - 4y^2$ by $-2x^2$.
Ans. $-24x^4 + 8x^2y^2$.

16. Multiply $x^3 + x^2y + xy^2 + y^3$ by $x - y$.
Ans. $x^4 - y^4$.

17. Multiply $x^2 + xy + y^2$ by $x^2 - xy + y^2$.
Ans. $x^4 + x^2y^2 + y^4$.

18. Multiply $3x^2 - 2xy + 5$ by $x^2 + 2xy - 3$.
Ans. $3x^4 + 4x^3y - 4x^2 - 4x^2y^2 + 16xy - 15$.

19. Multiply $3x^3 + 2x^2y^2 + 3y^3$ by $2x^3 - 3x^2y^2 + 5y^3$.
Ans. $6x^6 - 5x^5y^2 + 21x^3y^3 - 6x^4y^4 + x^2y^5 + 15y^6$.

IV. *Division, Art.* 46.

20. Divide $10x^2y - 15y^2 - 5y$ by $5y$.
Ans. $2x^2 - 3y - 1$.

21. Divide $3a^2 - 15 + 6a + 3b$ by $3a$.
Ans. $a - \frac{5}{a} + 2 + \frac{b}{a}$.

22. Divide $6x^4 - 96$ by $3x - 6$.
Ans. $2x^3 + 4x^2 + 8x + 16$.

23. Divide $48x^3 - 76ax^2 - 64a^2x + 105a^3$ by $2x - 3a$.
Ans. $24x^2 - 2ax - 35a^2$.

V. *Reduction of Fractions, Art.* 50 *and* 52.

24. What is the greatest common measure of $\frac{cx + x^2}{ca^2 + a^2x}$?
Ans. $c + x$.

25. What is the greatest common measure of $\frac{x^2 - 1}{xy + y}$?

Ans. $x + 1$.

26. What is the greatest common divisor of $\frac{x^2 - y^2}{x^4 - y^4}$?

Ans. $x^2 - y^2$.

27. Reduce $\frac{x^4 - b^4}{x^5 - b^2 x^3}$ to its lowest terms.

Ans. $x^2 - b^2$ gr. c. d. and $\frac{x^2 + b^2}{x^3}$ lowest terms.

28. Reduce $\frac{5a^5 + 10 a^4 x + 5 a^3 x^2}{a^3 x + 2 a^2 x^2 + 2 a x^3 + x^4}$ to its lowest terms.

Ans. $a + x$ gr. c. d. and $\frac{5 a^4 + 5 a^3 x}{a^2 x + a x^2 + x^3}$ lowest terms.

29. Reduce $\frac{3}{4}$, $\frac{2x}{3}$ and $a + \frac{2x}{a}$ to equivalent fractions having a common denominator.

Ans. $\frac{9a}{12a}$, $\frac{8ax}{12a}$ and $\frac{12a^2 + 24x}{12a}$.

30. Reduce $\frac{1}{2}$, $\frac{a^2}{3}$ and $\frac{x^2 + a^2}{x + a}$ to fractions having a common denominator.

Ans. $\frac{3x + 3a}{6x + 6a}$, $\frac{2a^2 x + 2a^3}{6x + 6a}$, and $\frac{6x^2 + 6a^2}{6x + 6a}$.

31. Reduce $\frac{b}{2a^2}$, $\frac{c}{2a}$ and $\frac{d}{a}$ to fractions having a common denominator.

Ans. $\frac{2a^2 b}{4a^4}$, $\frac{2a^3 c}{4a^4}$ and $\frac{4a^3 d}{4a^4}$.

VI. *Multiplication and Division of Fractions, Art.* 51.

32. What is the product of $\frac{x}{a}$ and $\frac{x + a}{a + c}$?

Ans. $\frac{x^2 + ax}{a + ac}$.

33. What is the product of $\frac{2x}{a}$, $\frac{3ab}{c}$ and $\frac{3ac}{2b}$?

Ans. $9ax$.

34. What is the product of $\frac{x^2-b^2}{bc}$ and $\frac{x^2+b^2}{b+c}$?

Ans. $\frac{x^4-b^4}{b^2c+bc^2}$.

35. What is the quotient of $\frac{x}{3}$ divided by $\frac{2x}{9}$?

Ans. $1\frac{1}{2}$.

36. What is the quotient of $\frac{x+1}{6}$ divided by $\frac{2x}{3}$?

Ans. $\frac{x+1}{4x}$.

37. What is the quotient of $\frac{x^4-b^4}{x^2-2bx+b^2}$ divided by $\frac{x^2+bx}{x-b}$?

Ans. $x+\frac{b^2}{x}$.

VII. *Addition and Subtraction of Fractions, Art.* 53.

38. Add $x+\frac{x-2}{3}$ to $3x+\frac{2x-3}{4}$.

Ans. $4x+\frac{10x-17}{12}$.

39. Add $\frac{2x}{3}$, $\frac{7x}{4}$, and $\frac{2x+1}{5}$ together.

Ans. $\frac{169x+12}{60}$ or $2x+\frac{49x}{60}+\frac{1}{5}$.

40. Add together $4x$, $\frac{7x}{9}$, and $2+\frac{x}{5}$.

Ans. $\frac{158x}{45}$ or $3x+\frac{23x}{45}$.

41. From $\frac{x+a}{b}$ subtract $\frac{c}{d}$. *Ans.* $\frac{dx+ad-bc}{bd}$.

42. From $\frac{3x}{7}$ subtract $\frac{2x}{9}$. *Ans.* $\frac{13x}{63}$.

43. From $3x+\frac{x}{b}$ subtract $x-\frac{x-a}{c}$.

Ans. $2x+\frac{cx+bx-ab}{bc}$.

VIII. *Problems in Simple Equations, Art.* 82.

44. In $5x - 15 = 2x + 6$ to find the value of x.

Ans. $x = 7$.

45. In $3y - 2 + 24 = 31$ to find y.

Ans. $y = 3$.

46. In the equations $\frac{x+2}{3} + 8y = 31$ and $\frac{y+5}{4} + 10x = 192$ to find x and y.

Ans. $x = 19$ and $y = 3$.

47. Out of a cask of wine, which had leaked away one third, 21 gallons were drawn, and then being gauged it was found to be half full: how much did it hold? *Ans.* 126 gallons.

48. What two numbers are those, whose difference is 7 and sum 33? *Ans.* 13 and 20.

49. What number is that from which if 5 be subtracted, two thirds of the remainder will be 40? *Ans.* 65.

50. At a certain election 375 persons voted, and the candidate chosen had a majority of 91 votes: how many voted for each candidate? *Ans.* 233 for one, and 142 for the other.

51. A post is $\frac{1}{4}$ in the mud, $\frac{1}{3}$ the water, and 10 feet above the water: what is its whole length? *Ans.* 24 feet.

52. A man arriving at Paris, spent the first day $\frac{1}{3}$ of the money he brought with him, the second day $\frac{1}{4}$, and the third day $\frac{1}{5}$, after which he had only 26 crowns left: how much did he have on arriving at Paris? *Ans.* 120 crowns.

53. A horse said to a mule, if I give you one of my sacks we shall be equally loaded, if I take one of yours I shall have twice as much as you: how many sacks had each?

Ans. The horse 7 and the mule 5.

54. A man being asked how many crowns he had, replied, if you add together a half, a third, and a quarter of what I have, the sum will exceed the number of crowns I have by one: what was the number? *Ans.* 12.

55. A privateer running at the rate of 10 miles an hour discovers a ship 18 miles off making way at the rate of 8 miles an hour: how many miles can the ship run before being overtaken?

Ans. 72 miles, or 9 hours.